CONTROLLABILITY TO SYSTEMS OF QUASILINEAR WAVE EQUATIONS WITH FEWER CONTROLS

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ABSTRACT

This study addresses challenges in solving exact boundary controllability problems of quasilinear wave equations with Dirichlet type boundary conditions satisfying some kind of Kalman rank conditions. Utilizing techniques such as characteristic method and a specially designed linearization iteration scheme, we developed controls that use fewer boundary controls than the classical theory.

INTRODUCTION

Controllability of Wave Systems

Consider the controllability for system of the quasilinear wave equations below

$$\begin{cases} U_{tt} - a^2(U, U_t, U_x)U_{xx} = 0, (t, x) \in \mathbb{R}^+ \times [0, L], \\ t = 0: (U, U_t) = (\varphi(x), \psi(x)), x \in [0, L], \\ x = L: U = 0, t \in \mathbb{R}^+, \\ x = 0: U_t - D(U, U_t, U_x)U_x = H(t), t \in \mathbb{R}^+, \\ t = T: (U, U_t) = (\Phi(x), \Psi(x)), x \in [0, L], \end{cases}$$

where

- $U = (u_1, \cdots, u_n)^T(t, x) \in C^2$,
- $a(U, U_t, U_x) > 0$ is the C^1 uniform wave speed,
- $D(U, U_t, U_x) = \left(d_{ij}(U, U_t, U_x)\right)_{i,j=1}^n$ is an $n \times n$ matrix with C^1 regularity,
- Part of the components of $H(t) = (h_1(t), \cdots, h_n(t))^T$ would be chosen as boundary controls,
- $\|(\phi(x), \psi(x))\|_{(C^2[0,L])^n \times (C^1[0,L])^n} \ll 1$,
- $\|(\Phi(x), \Psi(x))\|_{(C^2[0,L])^n \times (C^1[0,L])^n} \ll 1$

Kalman Rank Condition

We define coupling matrix and controlling matrix as

$$A = -D^-(0,0,0)D^+(0,0,0)^{-1},$$

$$B = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

then the Kalman type condition as

$$\mathrm{Rank}[B, AB, \cdots, A^{k-1}B] = n.$$

We try to utilize this condition to use controls fewer than classical results to achieve exact controllability.

Characteristic Formulation

Set

$$D^\pm(U, U_t, U_x) = aI_n \pm D,$$

$$\tilde{U} = D^+(0,0,0)U,$$

$$\tilde{V}^\pm = \partial_t \tilde{U} \mp \tilde{a}(\tilde{U}, \tilde{U}_t, \tilde{U}_x)\partial_x \tilde{U},$$

where we request $D^\pm$ are invertible. The original equations can be changed into

$$\begin{cases} \partial_t \tilde{V}^\pm \pm \tilde{a}(\tilde{U}, \tilde{U}_t, \tilde{U}_x)\partial_x \tilde{V}^\pm = \mp \partial_x \tilde{U}(\partial_t \pm \tilde{a}\partial x)\tilde{a}, \\ x = L: \tilde{V}^- = -\tilde{V}^+, \\ x = 0: \tilde{V}^+ = A\tilde{V}^- + 2\tilde{a}H(t) + Q(\tilde{U}, \tilde{U}_t, \tilde{U}_x). \end{cases}$$

where $Q(\tilde{U}, \tilde{U}_t, \tilde{U}_x)$ is a second order term. Thus the controllability for the wave equations is transformed into the one for a hyperbolic system of equations.

MAIN STEP 1

linearization Iteration Scheme:

- For the quasilinear equations, we start by proving an iterative proposition.

  For the iteration

$$U^{(0)} \equiv 0,$$

$$\begin{cases} \partial_t \tilde{V}^{\pm(m+1)} \pm \tilde{a}^{(m)} \partial_x \tilde{V}^{\pm(m+1)} = \mp \partial_x \tilde{U}^{(m)} (\partial_t \pm \tilde{a}^{(m)} \partial x) \tilde{a}^{(m)}, \\ t = 0: \left(U^{(m+1)}, U_t^{(m+1)}\right) = (\varphi(x), \psi(x)), x \in [0, L], \\ x = L: \tilde{V}^{-(m+1)} = -\tilde{V}^{+(m+1)}, \\ x = 0: \tilde{V}^{+(m+1)} = A\tilde{V}^{-(m+1)} + H^{(m+1)}(t) + Q^{(m)}, \\ t = T: \left(U^{(m+1)}, U_t^{(m+1)}\right) = (\Phi(x), \Psi(x)), x \in [0, L], \end{cases}$$

one can get uniform C^2 bounds and Cauchy sequence in C^1

$$\|U^{(m)}\|_{C^2} \le C\varepsilon$$

$$\|U^{(m)} - U^{(m-1)}\|_{C^1} \le C\varepsilon\alpha^{m-1}$$

with additional C^2-equi-continuity by estimates on modulus of continuity

$$\omega\left(\eta \middle| \frac{\partial^2 U^{(m)}}{\partial t^2}\right) + \omega\left(\eta \middle| \frac{\partial^2 U^{(m)}}{\partial x^2}\right) + \omega\left(\eta \middle| \frac{\partial^2 U^{(m)}}{\partial t \partial x}\right) \le \Omega(\eta).$$

- The sequence $\{U^{(m)}\}_{m=1}^\infty$ is a Cauchy sequence in C^1 space, and thus converges to some C^1 function U uniformly, applying Arzelà–Ascoli theorem, we know that there exists a subsequence of $\{U^{(m)}\}_{m=1}^\infty$,which converges uniformly in C^2 space. Thus we know that the whole original sequence $\{U^{(m)}\}_{m=1}^\infty$ converges to U in C^2 space. Therefore, U is C^2 smooth and satisfies the control condition.

MAIN STEP 2

Characteristic Method:

- In order for clarity of the use of Kalman rank condition in linear problems, we consider the following simple situation:

$$\begin{cases} \partial_t u_r - \partial_x u_r = 0, r = 1, \cdots, m, \\ \partial_t u_s + \partial_x u_s = 0, s = m+1, \cdots, n. \end{cases}$$

The boundary conditions are

$$\begin{cases} x = 0: \ u_s = \sum_{r=1}^m a_{sr} u_r + H_s(t), s = m+1, \cdots, n, \\ x = L: \ u_r = \sum_{s=m+1}^n a_{rs} u_s + H_r(t), r = 1, \cdots, m. \end{cases}$$

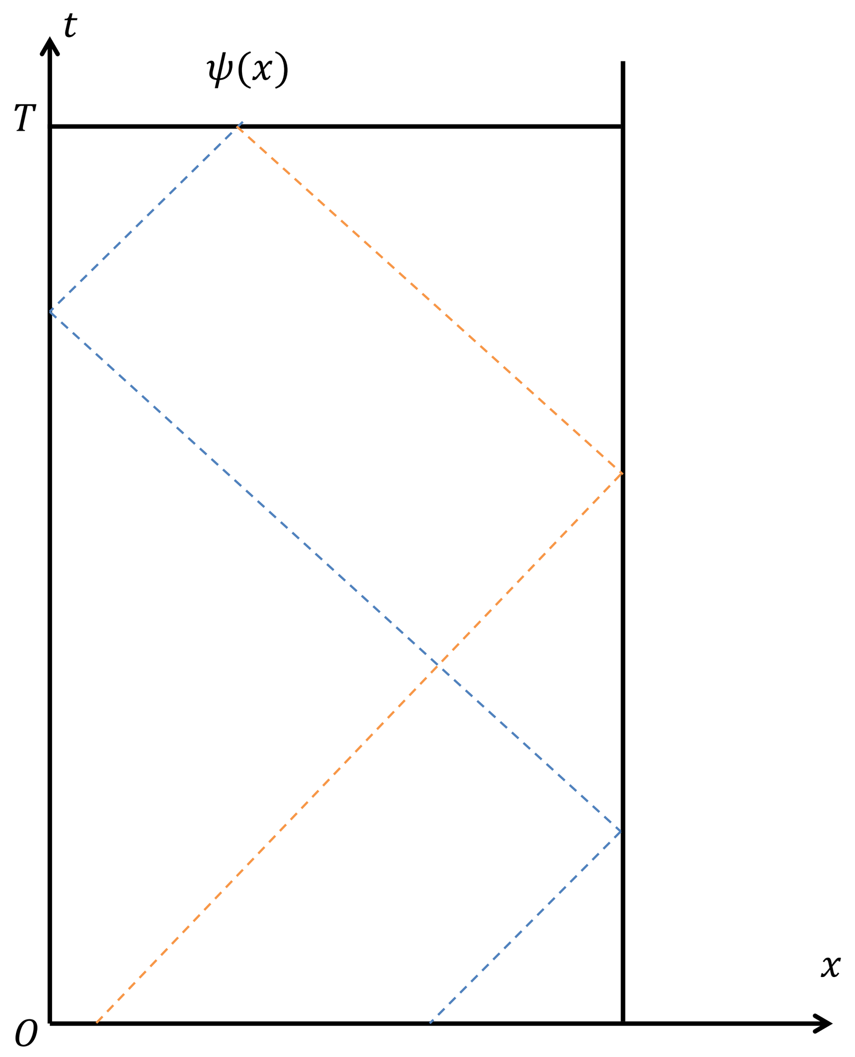

Figure 1: Characteristic Lines Converge to (T, x)

- We can obtain the following equation when $\varphi = 0$ with characteristic method

$$\begin{aligned} \vec{\psi}(x) &= \vec{u}(T, x) \\ &= \left(\vec{u}_r(T - L + x), \vec{u}_s(T - x)\right)^T \\ &= A\left(\vec{u}_r(T - x - L), \vec{u}_s(T - 2L + x)\right)^T + B\vec{H}_1(x) \\ &\cdots \\ &= A^k B\vec{H}_{k+1}(x) + \cdots + AB\vec{H}_2(x) + B\vec{H}_1(x) \end{aligned}$$

Where $\vec{H}_l(x) = \left(\vec{H}_r(T - x - (l-1)L), \vec{H}_s(T + x - lL)\right)^T$.

Denote $A = (a_{ij})_{i,j=1}^n$.

- We can perceive that the coefficient of the controls is

$$[B, AB, \cdots, A^{k-1}B]$$

Utilizing Kalman rank conditions one can obtain that a column of controls can be taken to make the system controllable.

- For the following linear equation

$$\begin{cases} \partial_t u_r - a(t, x)\partial_x u_r = F_r(t, x), r = 1, \cdots, m, \\ \partial_t u_s + a(t, x)\partial_x u_s = F_s(t, x), s = m+1, \cdots, n. \end{cases}$$

The boundary conditions are

$$\begin{cases} x = 0: \ u_s = \sum_{r=1}^m a_{sr} u_r + G_s(t) + H_s(t), s = m+1, \cdots, n, \\ x = L: \ u_r = \sum_{s=m+1}^n a_{rs} u_s + G_r(t) + H_r(t), r = 1, \cdots, m. \end{cases}$$

We can also obtain the controllability with similar process. Based on the procedure of calculation it can be obtained that the controls of the equations above satisfies the following paradigm estimate:

$$\|H\|_{C^1} \le C_A C_k (\|\varphi\|_{C^1} + \|\psi\|_{C^1} + \|G\|_{C^1} + \|F\|_{C^1})$$

CONCLUSION

Through the application of the characteristic method and the exploration of Kalman rank conditions, we have established a new framework for controlling quasilinear wave equations under a variety of conditions.

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