Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Chair
DYNAMICS, CONTROL
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FAU

Unterstützt von / Supported by
Alexander von Humboldt
Stiftung / Foundation

NODAL CONTROL AND THE TURNPIKE PHENOMENON

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TRR 154 - Mathematical Modelling, Simulation and Optimization using the Example of Gas Networks (Subproject C03)

TRR 154
The "turnaround in energy policy" is currently in the main focus of public opinion. It concerns social, political and scientific aspects as the dependence on a reliable, efficient and affordable energy supply becomes increasingly dominant. On the other side, the desire for a clean, environmentally consistent and climate-friendly energy production is stronger than ever.

Natural gas, hydrogen and other energy sources:
To balance these tendencies while making a transition to nuclear-free energy supply, natural gas is an important energy source in the current decade. Hydrogen as energy source will play an important role in the near future. Hydrogen is an energy carrier and can deliver or store a tremendous amount of energy. Gaseous hydrogen can be transported through pipelines much the way natural gas is today. The mathematical results in this context are also applicable in other context, e.g., to physical transport systems like water networks.

Mathematical Models for the Gas Flow in Pipelines

ISO1 $\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha) \end{cases}$

ISO3 $\begin{cases} \rho_t + (\rho v)_x = 0 \\ p_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha) \end{cases}$

ISO2 $\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + p_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha) \end{cases}$

ISO4 $\begin{cases} (\rho v)_x = 0 \\ p_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha) \end{cases}$

p	gas pressure	v	gas velocity	g	gravitational constant
ρ	gas density	λ/D	pipe friction	α	pipe slope

Mathematical Modelling on Networks

Isothermal Euler Equations

$$\rho_t + (\rho v)_x = 0$$
$$(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$

$v_0 \to v_1 \to \otimes \to v_3 \to v_4$

Source: Inlet Pressure
$p(t) = p_0(t)$

Compressor Station
$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$

Sink: Gas Outflow
$(\rho v)(t) = b(t)$

Coupling Conditions

Conservation of Mass: $\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t)$, Continuity in Pressure: $p_{\text{in}} = p_{\text{out}}$

Optimal Compressor Control with Buffer Zones on Gas Networks

Let bounds for the pressures $0 < p_{\min} < p_{\max}$ be given at every node. For $\varepsilon > 0$ consider the optimal control problem

$$\min_{u \in L^2(0,T)} f(u)$$

s.t.

$\rho_t + (\rho v)_x = 0$, $(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v\lvert v\rvert - g\rho\sin(\alpha)$	on every edge
$p(t) = p_0(t)$	on every source node
$(\rho v)(t) = b(t)$	on every sink node
$\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t)$, $p_{\text{in}} = p_{\text{out}}$	on every inner node
$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$	for every compressor
$p(t) \in \left[\, p_{\min} + \varepsilon, p_{\max} - \varepsilon \,\right] \quad \forall t \in [0,T]$	on every node

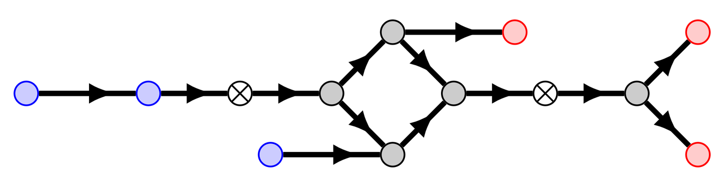
Uncertainty in the boundary data

- For an optimal control u^* we a posteriori assume, that the boundary data is random, i.e. we assume $b(t) = b^\omega(t)$, where $b^\omega(t)$ is given by a randomized Fourier series of $b(t)$
- Our aim is to compute the probability, that the pressure at the nodes corresponding to the random gas demand stays within the pressure bounds, i.e., for $\varepsilon > 0$ with corresponding optimal control $u^*(t)$ we compute the probability

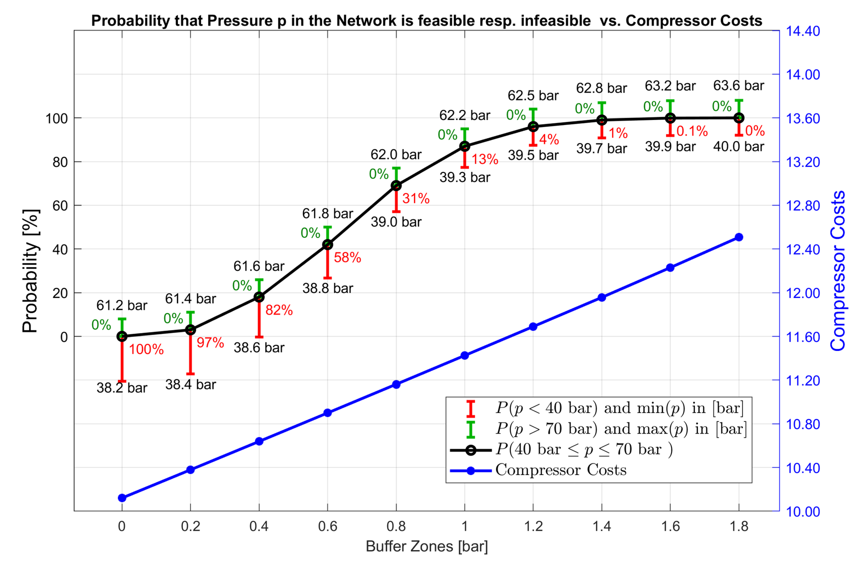
$$\mathbb{P}\Big(p^\omega(t, u^*(t)) \in \left[\, p^{\min}, p^{\max} \,\right] \quad \forall t \in [0,T] \Big).$$

Probabilistic Robustness

- Application to real network instance with data close to reality from *https://gaslib.zib.de/*:



- Stochastic collocation increases speed of computation enormously
- Kernel density estimation provides information on the probability density function and allows to compute derivatives of the probability



The Turnpike Phenomenon under Uncertainty

- There is a fastest route between any two points and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
  But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.
- For convex functions f and g, consider the optimal control problems

$$\begin{cases} \min\limits_{u \in L^2(0,T)} & J_T(u) = \int_0^T f(u(t)) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} & r_t(t,x) + c\, r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{cases} \qquad \begin{cases} \min\limits_{u \in \mathbb{R}} & J(u) = f(u) + g\big(r(L)\big) \\ \text{s.t.} & c\, r_x(x) = m\, r(x), \\ & r(0) = u. \end{cases}$$

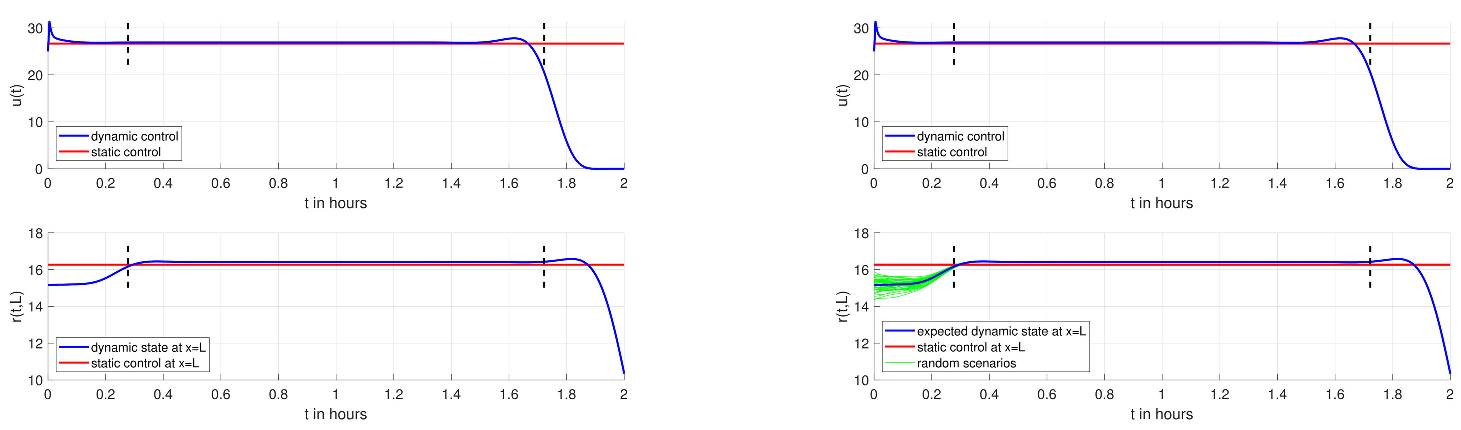
- If f is strongly convex and if g' is Lipschitz continuous, then for the optimal controls $u^\delta(t) \in L^2(0,T)$ and $u^\sigma \in \mathbb{R}$ we have

$$\int_0^T \left\| u^\delta(t) - u^\sigma \right\|_2^2 \, dt \le \mathcal{C}. \tag{1}$$

- Assume that the initial data is perturbed by a *Wiener process*, i.e., we have $r^\omega_{\text{ini}} = r_{\text{ini}} + W_x$, and consider the objective function

$$J_T(u) = \int_0^T f\big(u(t)\big) + g\Big(\mathbb{E}\big[r(t,L)\big]\Big)\, dt.$$

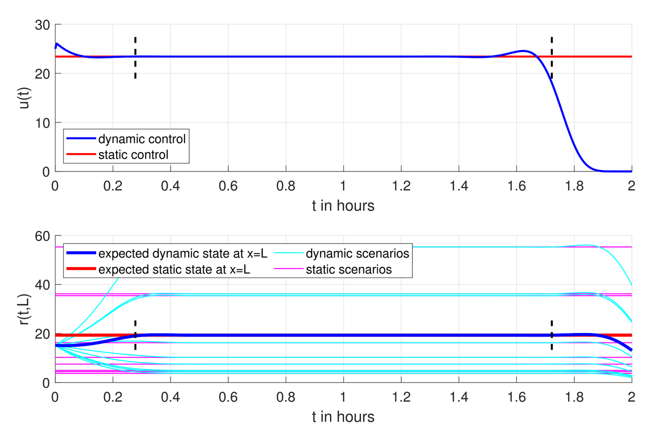
Then the turnpike inequality (1) holds as well.



- Consider the optimal control problem with random source term

$$\begin{cases} \min\limits_{u \in L^2(0,T)} & \int_0^T f\big(u(t)\big) + g\Big(\mathbb{E}\big[r(t,L)\big]\Big)\, dt \\ \text{s.t.} & r_t(t,x) + c\, r_x(t,x) = m^\omega\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{cases}$$

- If the expected value $\mathbb{E}\big[r(t,L)\big]$ exists, then the turnpike inequality (1) holds as well



- Extension of the theory to linear hyperbolic 2×2 systems possible via adjoint calculus
- Simulations also show a turnpike properties for nonlinear equations and systems

Selected publications

[1] Schuster, M. (2021). **Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks.** Dissertation, FAU Erlangen-Nürnberg, Germany, available under `https://opus4.kobv.de/opus4-trr154/home`

[2] Gugat, M., Schuster, M., Zuazua, E. (2021). **The Finite-Time Turnpike Phenomenon for Optimal Control Problems: Stabilizytion by Non-Smooth Tracking Terms.** Stabilization of Distributed Parameter Systems: Design Methods and Applications.

[3] Schuster, M., Strauch, E., Gugat, M., Lang, J. (2022). **Probabilistic Constrained Optimization on Flow Networks.** Optim. Eng. 23, 1–50.

[4] Sakamoto, N., Schuster, M. (2024). **A Turnpike Result for Boundary Control Problems with the Transport Equation under Uncertainty.** Submitted, Preprint available at `https://opus4.kobv.de/opus4-trr154/home`

TRR 154

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