Global Entropy Solutions to Isentropic Gas Flows in General Nozzles

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Introduction

- We consider isentropic gas flows in general nozzles, which are governed by the **quasi-one-dimensional isentropic Euler system**:

$$(1)\quad \begin{cases} \partial_t\rho + \partial_x(\rho v) = a(x)\rho v, \\ \partial_t(\rho v) + \partial_x\big(\rho v^2 + p(\rho)\big) = a(x)\rho v^2, \\ t = 0: \rho = \rho_0(x), v = v_0(x). \end{cases}$$

- In engineering, nozzles are useful in various areas, such as the modern rocket engine or the jet engine.
- In astrophysics, it is known that the plasma flow from stars is closely related to the spherically symmetric flow.
- From a mathematical point of view, the system is a typical quasi-linear hyperbolic balance law system.
- **Our aim is to prove the global existence of its entropy weak solutions with large data.**

Motivations

- We focus on the case where $a(x) \in L^1(\mathbb{R})$. In order to give the existence theorem, previous articles have proposed upper bound estimates:

  a) In [1], it's required

$$\|a(x)\|_{L^1} \le \frac{1-\theta}{1+\theta}.$$

  b) In [2], it's required

$$\max\left\{\int_0^{\infty} |a(x)|\mathrm{d}x\,,\int_{-\infty}^{0} |a(x)|\mathrm{d}x\right\} \le \frac{\mu}{2}\ln\frac{1}{\sigma},$$

  where θ, μ, σ are kinetic constants.

- **Our work no longer sets any a priori upper bound on** $\|a(x)\|_{L^1}$.

Methods

- **Compensated compactness framework** is applied to complete the proof.
- The vanishing viscosity method is used to construct the approximate solution.
- New modified Riemann invariants and invariant regions are introduced.

Key points

- Artificial ε−viscosities are added to the δ−flux approximation system as

$$(2)\quad \begin{cases} \partial_t\rho + \partial_x(\rho v - 2\delta v) = a(x)(\rho - 2\delta)v + \varepsilon\partial_{xx}\rho, \\ \partial_t(\rho v) + \partial_x\big(\rho v^2 - 2\delta v^2 + \tilde{p}(\rho,\delta)\big) \\ \qquad\qquad = a(x)(\rho - 2\delta)v^2 + \varepsilon\partial_{xx}(\rho v), \\ t = 0: \rho = \rho_0(x) + 2\delta, v = v_0(x). \end{cases}$$

- Modified Riemann invariants are introduced as

$$\tilde{z} = z \cdot e^{-\int_{x_0}^{x} b(y)dy} - K_\delta,$$
$$\tilde{w} = w \cdot e^{-\int_{x_0}^{x} b(y)dy} - K_\delta.$$

- Riemann invariants stay in the invariant region.

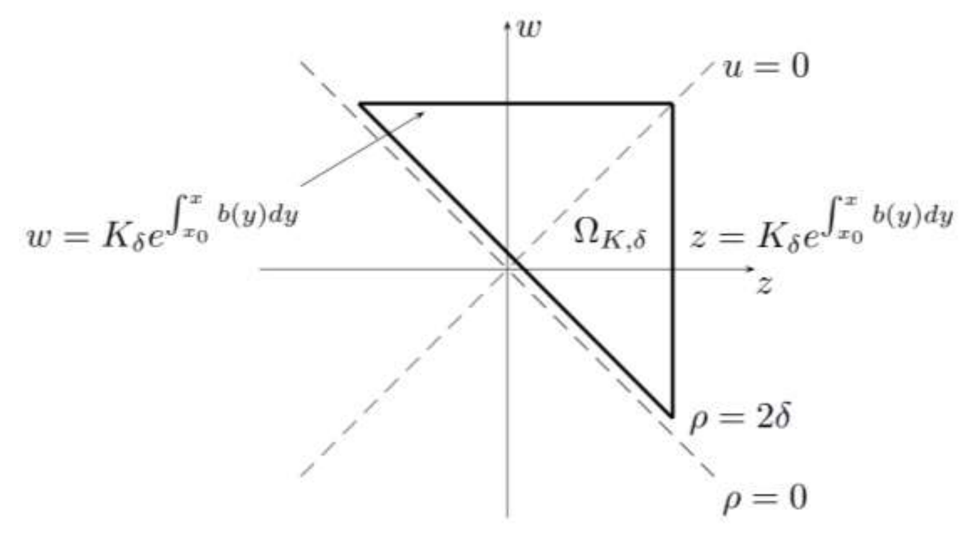

Fig. 1. Region $\Omega_{K,\delta}$

- The approximate system (2) shares the same entropy function with system (1). Furthermore, H^{-1} compactness estimate for corresponding entropy-entropy flux pair holds.

Conclusion

As long as

$$a(x) \in W^{1,\infty}(\mathbb{R}) \cap L^1(\mathbb{R}),$$

for any data $\rho_0, v_0 \in L^\infty(\mathbb{R})$, **the system (1) admits a global entropy weak solution.**

References

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2024 Conference on Control, Inversion and Numerics for PDEs
October 7th-10th, 2024, Fudan University