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Non-uniqueness for the compressible Euler-Maxwell equations

Shunkai Mao and Peng Qu

School of Mathematical Sciences, Fudan University, China.

Introduction

- We consider the isentropic compressible Euler-Maxwell system on the periodic domain $[0,T]\times\mathbb{T}^3$ with $\mathbb{T}^3=[-\pi,\pi]^3$ and $T\in(0,\infty)$. The Cauchy problem with initial condition can be expressed as follows:

$$\begin{cases}\partial_t n+\operatorname{div}\ m=0,\\ \partial_t m+\operatorname{div}\left(\dfrac{m\otimes m}{n}\right)+\nabla p(n)=-nE-m\times B,\\ \partial_t E-\nabla\times B=m,\quad \operatorname{div}E=h(x)-n,\\ \partial_t B+\nabla\times E=0,\quad \operatorname{div}B=0,\\ (n,m,E,B)|_{t=0}=(n_0,m_0,E_0,B_0).\end{cases}$$

- The Euler-Maxwell system is a hydrodynamic model used in plasma physics to describe the motion of electrons under the influence of the corresponding electromagnetic fields.
- We consider weak solutions (n,m,E,B) which are Hölder continuous in space, for instance, for some constant C which is independent of t,

$$|m(t,x)-m(t,y)|\leqslant C|x-y|^\alpha,\qquad \forall x,y\in\mathbb{T}^3,\forall t\in[0,T].$$

- We consider the entropy inequality as

$$\partial_t\left(\frac{|m|^2}{2n}+\frac{|E|^2+|B|^2}{2}+ne(n)\right)+\operatorname{div}\left(\frac{m}{n}\left(\frac{|m|^2}{2n}+ne(n)+p(n)\right)+E\times B\right)\leqslant 0.$$

Induction scheme

For a given $n=n(t,x)\in C^\infty([T_1,T_2]\times\mathbb{T}^3)$, $h=h(x)\in C^\infty(\mathbb{T}^3)$ with $n(t,x)\geqslant\varepsilon_0$ for some positive constant ε_0, and $\fint_{\mathbb{T}^3}n(t,x)\mathrm{d}x=\fint_{\mathbb{T}^3}h(x)\mathrm{d}x$ for all t, a tuple of smooth tensors (m,E,B,c,R,φ) is a dissipative Euler-Maxwell-Reynolds flow as long as it solves the following system

$$\begin{cases}\partial_t n+\operatorname{div}m=0,\\ \partial_t m+\operatorname{div}\left(\dfrac{m\otimes m}{n}\right)+\nabla p(n)+nE+m\times B=\operatorname{div}(n(R-c\,\mathrm{Id})),\\ \partial_t E-\nabla\times B=m,\quad \operatorname{div}E=h(x)-n,\\ \partial_t B+\nabla\times E=0,\quad \operatorname{div}B=0,\\ \partial_t\left(\dfrac{|m|^2}{2n}+\dfrac{|E|^2+|B|^2}{2}+ne(n)\right)\\ \quad+\operatorname{div}\left(\dfrac{m}{n}\left(\dfrac{|m|^2}{2n}+ne(n)+p(n)\right)+E\times B\right)\\ \quad=n\left(\partial_t+\dfrac{m}{n}\cdot\nabla\right)\dfrac12\operatorname{tr}(R)+\operatorname{div}((R-c\,\mathrm{Id})m)+\operatorname{div}(n\varphi)+\partial_t H.\end{cases}$$

Here H is the global energy loss, which satisfies $H(0)=0,\partial_t H\leqslant 0$.

- We construct a sequence of approximate solutions (dissipative Euler-Maxwell-Reynolds flows), which converge to a weak solution of the Euler-Maxwell system.
- At each step, we give a correction $(\tilde m_q,\tilde E_q,\tilde B_q)=(m_{q+1}-m_q,E_{q+1}-E_q,B_{q+1}-B_q)$ to make the error (R_q,φ_q) get smaller which would converge to zero (in Hölder space) as q goes to infinity.
- We could obtain the inductive estimates on $(m_q,E_q,B_q,R_q,\varphi_q)$.

Main Results

We prove two main theorems that imply the non-uniqueness of entropy solutions in the Hölder class $C^{\frac17-}$ to the compressible Euler-Maxwell equations.

Theorem 1. *For any $0\leqslant\beta<1/7$, initial density $n_0=n_0(x)\in C^\infty(\mathbb{T}^3)$, $h=h(x)\in C^\infty(\mathbb{T}^3)$, and pressure $p=p(n)\in C^\infty([\varepsilon_0,\infty))$, where $\fint_{\mathbb{T}^3}n_0(x)\mathrm{d}x=\fint_{\mathbb{T}^3}h(x)\mathrm{d}x$, and ε_0 is a positive constant such that $n_0(x)\geqslant\varepsilon_0$, we can find infinitely many distinct entropy solutions, $n\in C^\infty([0,T]\times\mathbb{T}^3)$, $m\in C^\beta([0,T]\times\mathbb{T}^3)$, $E\in C^{1,\beta}([0,T]\times\mathbb{T}^3)$, and $B\in C^{1,\beta}([0,T]\times\mathbb{T}^3)$, to the isentropic compressible Euler-Maxwell equations emanating from the same initial data and satisfying the energy equation in the distributional sense.*

Theorem 2. *For any $0\leqslant\beta<1/7$, initial density $n_0=n_0(x)\in C^\infty(\mathbb{T}^3)$, $h=h(x)\in C^\infty(\mathbb{T}^3)$, and pressure $p=p(n)\in C^\infty([\varepsilon_0,\infty))$, where $\fint_{\mathbb{T}^3}n_0(x)\mathrm{d}x=\fint_{\mathbb{T}^3}h(x)\mathrm{d}x$, and ε_0 is a positive constant such that $n_0(x)\geqslant\varepsilon_0$, there is an entropy solution $n\in C^\infty([0,T]\times\mathbb{T}^3)$, $m\in C^\beta([0,T]\times\mathbb{T}^3)$, $E\in C^{1,\beta}([0,T]\times\mathbb{T}^3)$, and $B\in C^{1,\beta}([0,T]\times\mathbb{T}^3)$, to the isentropic compressible Euler-Maxwell equations satisfying the entropy inequality strictly in the distributional sense.*

Methods and Challenges of the Proof

- The proof of our results relies on the convex integration scheme starting from De Lellis-Székelyhidi. We adapt the convex integration scheme proposed by De Lellis-Kwon [1] and Giri-Kwon [2] to the compressible Euler-Maxwell system.
- We propose a new method of Mikado potential. We use the specially chosen electromagnetic potentials to construct new building blocks $(\mathring{m}_k,\mathring{E}_k,\mathring{B}_k)$ which satisfying the Maxwell equations and can be used to construct the perturbation. In this way, we can not only express the solutions of the Maxwell equations explicitly, but also use a special linear combination of the main part of $\mathring{m}_k$, denoted by $\mathring{m}_{p,k}$, to construct the Mikado flows.
- Due to the constrain of the Maxwell equations and strong resonance between the electromagnetic fields may occur, we would find that for some directions, a strong electromagnetic field can only lead to a weak fluid flow. Then, the special type of Mikado flows will lose certain frequencies, that is, the terms corresponding to certain frequency will be close to zero. If we use the special Mikado potentials to construct $\tilde m_p$, the low-frequency components of $\frac{\tilde m_p\otimes\tilde m_p}{n}$ and $\frac{|\tilde m_p|\tilde m_p}{2n^2}$ may vanish. To solve this, we would specially choose the strength function ψ^*, which allows us to use the low-frequency components of $\frac{\tilde m_p\otimes\tilde m_p}{n}$ and $\frac{|\tilde m_p|\tilde m_p}{2n^2}$ to reduce the Reynolds error R_q and current φ_q separately.

References

[1] C. De Lellis and H. Kwon. On nonuniqueness of Hölder continuous globally dissipative Euler flows. *Anal. PDE*, 15(8):2003–2059, 2022.

[2] V. Giri, and H. Kwon. On non-uniqueness of continuous entropy solutions to the isentropic compressible Euler equations. *Arch. Ration. Mech. Anal.*, 245(2):1213–1283, 2022.

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