Stability analysis of steady transonic flows with an external force in a three-dimensional straight

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Introduction

This research mainly focuses on the well-posedness of two types of transonic flows in three-dimensional straight nozzles under the influence of external forces:

1. The smooth axisymmetric transonic irrotational flow in cylindrical nozzles, *arXiv:2401.05029*;
2. The three-dimensional transonic shock wave in rectangular nozzles, *arXiv:2312.02688.*

Equations

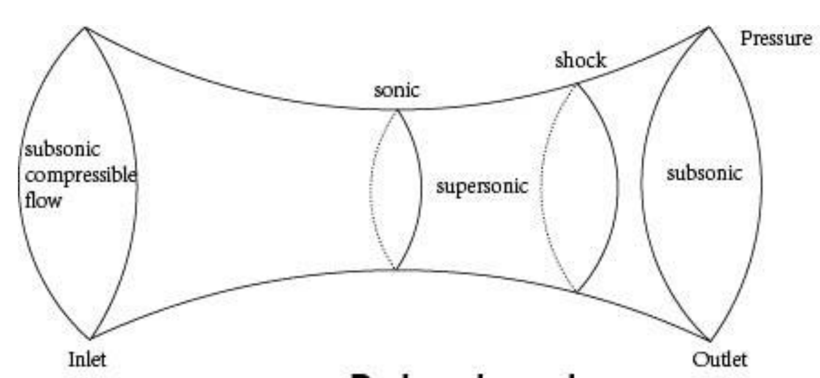
The steady three dimensional isentropic Euler equations with an external force:

$$\begin{cases} \mathbf{div}\,(\rho \boldsymbol{u}) = \mathbf{0}, \\ \mathbf{div}\,(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathrm{P} \boldsymbol{I_3}) = \boldsymbol{\rho \nabla \Phi} \end{cases}$$

$\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, ρ is the density, $P = A\rho^\gamma$ $(\gamma > 1)$ is the pressure and Φ is the potential force.

Motivation

In the design of engines, a key component is the de Laval nozzle, devised by Swedish engineer Carl Gustaf Patrik de Laval, which features a geometry that contracts and then expands. Due to the geometric effect of the nozzle, the subsonic flow accelerates in the convergent section, forming a smooth transonic flow; the appropriately large pressure at the outlet generates transonic shock waves in the divergent section. We investigate the existence of smooth transonic flows and transonic shock waves under the affection of external forces.



De Laval nozzle

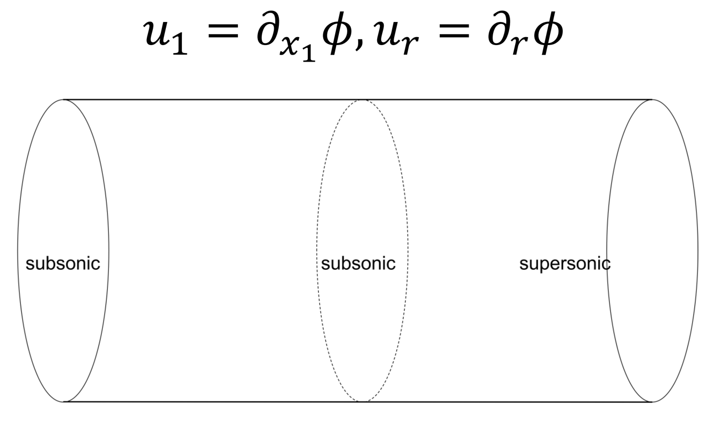
One-dimensional transonic flow under external force

Smooth axisymmetric transonic irrotational flows

Result

Concentrate only on the irrotational flows: $\omega_\theta = \partial_{x_1} u_r - \partial_r u_1 = 0$ in the cylinder and introduce the potential function ϕ, such that

$$u_1 = \partial_{x_1}\phi, u_r = \partial_r \phi$$



Cylinder

In this cylinder, there exists **a unique smooth axisymmetric irrotational solution** $\phi \in H_r^4$ in the steady Euler system with external forces, which exhibits structural stability with respect to the one-dimensional background solution under perturbations of appropriate boundary conditions, that is

$$\|\phi - \bar{\phi}\| \le C_0 \varepsilon$$

Key idea

- Some weighted Sobolev spaces $H_r^m(D)$ are introduced to deal with the singularities near the axis.
- The multiplier method and Galerkin's method are employed to prove the existence and uniqueness of a strong solution in $H_r^2(D)$, where the choice of multiplier strongly depends crucially on the properties of the one-dimensional background smooth transonic flow.
- Given that the equation will change its type, we analyze the regularity separately in subsonic and transonic regions.

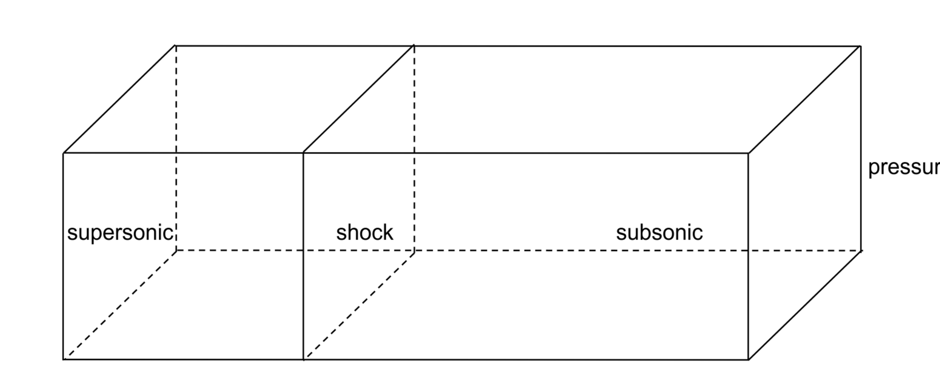
Reference

- S. WENG, Z. XIN. Smooth transonic flows with nonzero vorticity to a quasi two dimensional steady Euler flow model[J]. Archive for Rational Mechanics and Analysis, 2024, 248(3): 49.

Three-dimensional transonic shock flows

Result

The external force has a stabilization effect on the transonic shocks in flat nozzles and the transonic shock is completely free, we ***do not*** require it to pass through a fixed point.



Rectangular Cylinder

In a rectangular cylinder**, the existence and uniqueness of the transonic shock solution** for a steady isentropic Euler system with an external force are established, which is structural stability under the three-dimensional perturbations for the incoming supersonic flow, the exit pressure and the external force, that is

$$\|\Psi - \bar{\Psi}\|_{C^{2,\alpha}} \le C_* \varepsilon$$

Key idea

- To avoid the influence of sonic lines, we assume that the incoming flow at the nozzle inlet is supersonic.
- The deformation-curl decomposition method proposed by Weng-Xin is used to decouple the steady Euler equations with an external force.
- Decompose the Rankine-Hugoniot conditions, clarifying the boundary conditions for elliptic and hyperbolic quantities on the shock front.
- Introduce an appropriate coordinate system to transform the free boundary value problem into a fixed boundary value problem, design a suitable iterative scheme.

Reference

- S. WENG, Z. XIN. Existence and stability of cylindrical transonic shock solutions under three dimensional perturbations[J]. arXiv: 2304.02429, 2023.