Optimal control of critical wave equations

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Workshop on Control, Inversion and Numerics for PDEs
Erlangen-Shanghai

August 2nd 2023

Semilinear wave optimal control problem

$$\min_{(y,u)} \quad \ell(y,u)$$
$$\text{s.t.} \quad y'' - \Delta y + y^5 = u \qquad \text{on } (0,\mathrm{T}) \times \Omega,$$
$$\|u(t)\|_2 \le \omega(t) \quad \text{on } (0,\mathrm{T})$$

- bounded smooth domain $\Omega \subset \mathbb{R}^3$, distributed control
- homogeneous Dirichlet or Neumann boundary conditions
- initial values $(y_0, y_1) \in \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega) =: \mathcal{E}$ (energy space)
- constraint set $\mathcal{U}_{\mathrm{ad}} = \Big\{ u \colon (0,\mathrm{T}) \to \mathrm{L}^2(\Omega) \colon \|u(t)\|_2 \le \omega(t) \Big\}$

The state equation

- defocusing H^1_0-critical semilinear wave equation

$$y''(t) - \Delta y(t) + y^5(t) = u(t), \quad y[0] = (y_0, y_1)$$

- nonlinearity y^p: $1 < p < 5$ subcritical, $p > 5$ supercritical
- **defocusing** $\sim$ sign of nonlinearity $\sim$ global-in-time existence
- $\Omega = \mathbb{R}^3$: global existence well established [Gri92, SS95, Sog95]
- Ω bounded: Dirichlet [BLP08] ($u = 0$), [KSZ16], Neumann [BP09]
- here: consider **mild solutions** $(\longrightarrow \quad u \in \mathrm{L}^1(0, T; \mathrm{L}^2(\Omega)))$

The state equation

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- here: consider **mild solutions** $(\longrightarrow \quad u \in \mathrm{L}^1(0, \mathrm{T}; \mathrm{L}^2(\Omega)))$

Mild$^+$ solution

Definition

Say: $y[\cdot] \in \mathrm{C}([0,\mathsf{T}];\mathcal{E})$ is **mild$^+$ solution** of CWE iff

$$y[t] = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \int_0^t e^{\mathcal{A}(t-s)} \begin{pmatrix} 0 \\ u(s) - y^5(s) \end{pmatrix} \, \mathrm{d}s$$

and **additionally** $y \in \mathrm{L}^4(0,\mathsf{T};\mathrm{L}^{12}(\Omega))$.

- $\mathcal{A}$ first order system wave operator in $\mathcal{E} = \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)$
- mild solution with additional integrability properties [SS94]
- mild$^+$ solution satisfies $y^5 \in \mathrm{L}^1(0,\mathsf{T};\mathrm{L}^2(\Omega))$:

$$\mathrm{L}^4(0,\mathsf{T};\mathrm{L}^{12}(\Omega)) \cap \mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^6(\Omega)) \hookrightarrow \mathrm{L}^5(0,\mathsf{T};\mathrm{L}^{10}(\Omega))$$

mild$^+$ solution: properties

- equivalent to: weak solution + additional integrability (weak$^+$)
- associated (conserved) **energy** $\qquad (E(t) = E(0))$

$$E(t) = \frac{1}{2}\|y[t]\|_{\mathcal{E}}^2 + \frac{1}{6}\|y(t)\|_{\mathrm{L}^6(\Omega)}^6 - \int_0^t \big(u(s), y'(s)\big)_\Omega \, \mathrm{d}s$$

- **energy bound**:

$$\|y[\cdot]\|_{\mathrm{C}([0,T];\mathcal{E})}^2 + \|y\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\Omega))}^6 \lesssim \|(y_0, y_1)\|_{\mathcal{E}}^2 + \|y_0\|_{\mathrm{L}^6(\Omega)}^6 + \|u\|_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}^2 =: E_0$$

- mild$^+$ solution **unique**, if it exists

Local existence

Theorem

For every $u \in \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^2(\Omega))$*: exists **unique maximal*** mild$^+$ *solution* $y = y_u$ *to CWE, i.e., maximal time* $T^\bullet \in (0,\mathrm{T}]$*:*

$$y[\cdot] \in \mathrm{C}([0,T^\bullet);\mathcal{E}) \quad \text{and} \quad y \in \mathrm{L}^4_{loc}([0,T^\bullet);\mathrm{L}^{12}(\Omega))$$

and y *satisfies mild solution formula on* $[0,T^\bullet)$*.*

- in fact: CWE **well-posed** ($\longrightarrow$ smooth approximation!)
- important ingredient: (**Strichartz estimate**)

$$\|z\|_{\mathrm{L}^4(0,T;\mathrm{L}^{12}(\Omega))} \lesssim \|(z_0,z_1)\|_{\mathcal{E}} + \|f\|_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}$$

  for mild solution z of linear WE with data (f,z_0,z_1) [BSS09]

Unconditional global existence

Theorem

*For every $u \in \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^2(\Omega))$, the unique local solution $y = y_u$ to CWE exists **globally** on $[0,\mathrm{T}]$ and satisfies*

$$y[\cdot] \in \mathrm{C}([0,\mathrm{T}];\mathcal{E}) \quad \textit{and} \quad y \in \mathrm{L}^4(0,\mathrm{T};\mathrm{L}^{12}(\Omega)).$$

- standard contradiction: extend solution over maximal time $T^\bullet$
- observation: t^* such that $y \in \mathrm{L}^4(t^*,T^\bullet;\mathrm{L}^{12}(\Omega))$ sufficient:

$$\|y^5\|_{\mathrm{L}^1(t^*,T^\bullet;\mathrm{L}^2(\Omega))} \leq \underbrace{\|y\|^{\frac{4}{5}}_{\mathrm{L}^\infty(t^*,T^\bullet;\mathrm{L}^6(\Omega))}}_{\lesssim \text{ energy bound } E_0} \|y\|^{\frac{1}{5}}_{\mathrm{L}^4(t^*,T^\bullet;\mathrm{L}^{12}(\Omega))}$$

- **no** $\|y^5\|_{\mathrm{L}^1(t^*,T^\bullet;\mathrm{L}^2(\Omega))}$ bound from equation (power 5 **critical**)

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For every $u \in \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^2(\Omega))$*, the unique local solution* $y = y_u$ *to CWE exists* ***globally*** *on* $[0,\mathrm{T}]$ *and satisfies*

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What does not work (but in the subcritical case)

Strichartz estimate:

$$\|y\|_{\mathrm{L}^4(t,T^\bullet;\mathrm{L}^{12}(\Omega))} \lesssim \sqrt{E_0} + \|y^p\|_{\mathrm{L}^1(t,T^\bullet;\mathrm{L}^2(\Omega))}$$

Hölder: $(1 < p \leq 5)$

$$\|y^p\|_{\mathrm{L}^1(t,T^\bullet;\mathrm{L}^2(\Omega))} \leq \|y\|^{p-1}_{\mathrm{L}^4(t,T^\bullet;\mathrm{L}^{12}(\Omega))} \|y\|_{\mathrm{L}^{\frac{4}{5-p}}(t,T^\bullet;\mathrm{L}^{\frac{12}{7-p}}(\Omega))},$$

and $\frac{12}{7-p} < p+1$ iff $1 < p < 5$, so

$$\|y\|_{\mathrm{L}^{\frac{4}{5-p}}(t,T^\bullet;\mathrm{L}^{\frac{12}{7-p}}(\Omega))} \lesssim_\Omega \left(T^\bullet - t\right)^{\frac{5-p}{4}} \underbrace{\|y\|_{\mathrm{L}^\infty(t,T^\bullet;\mathrm{L}^{p+1}(\Omega))}}_{\lesssim \text{ energy bound } E_0}.$$

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$\longrightarrow$ exists $t^\star$: $\|y\|_{\mathrm{L}^4(t^\star,T^\bullet;\mathrm{L}^{12}(\Omega))}$ **bounded**!

- global existence for subcritical nonlinearity $1 < p < 5$
- unfortunate failure for $p = 5$: $\lim_{t \nearrow T^\bullet} \|y\|_{\mathrm{L}^\infty(t,T^\bullet;\mathrm{L}^6(\Omega))} = 0$?

Idea:

$$\|y\|_{\mathrm{L}^4(t,T^\bullet;\mathrm{L}^{12}(\Omega))} \lesssim \sqrt{E_0} + \|y\|_{\mathrm{L}^\infty(t,T^\bullet;\mathrm{L}^6(\Omega))}\|y\|^4_{\mathrm{L}^4(t,T^\bullet;\mathrm{L}^{12}(\Omega))}$$

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Idea: $(\Lambda(t, T^\bullet) \subset (0, T) \times \Omega)$

$$\|y\|_{L^4_t L^{12}_x(\Lambda(t,T^\bullet))} \lesssim \sqrt{E_0} + \underbrace{\|y\|_{L^\infty_t L^6_x(\Lambda(t,T^\bullet))}}_{\to 0 \text{ as } t \searrow T^\bullet} \|y\|^4_{L^4_t L^{12}_x(\Lambda(t,T^\bullet))}$$

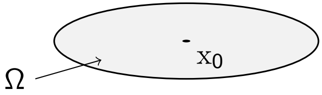
Proof idea/outline

> *"One of the problems has to do with the speed of light and the difficulties involved in trying to exceed it. You can't. Nothing travels faster than the speed of light with the possible exception of bad news, which obeys its own special laws."*
>
> Douglas Adams, *Mostly Harmless*

Proof idea/outline

time-space localize to **backwards light cone** from $(T^\bullet, \mathrm{x}_0)$:

$$\Lambda(s, T^\bullet) := \Big\{(t, \mathrm{x}) \in [s, T^\bullet] \times \overline{\Omega} \colon |\mathrm{x} - \mathrm{x}_0| < T^\bullet - t\Big\}$$



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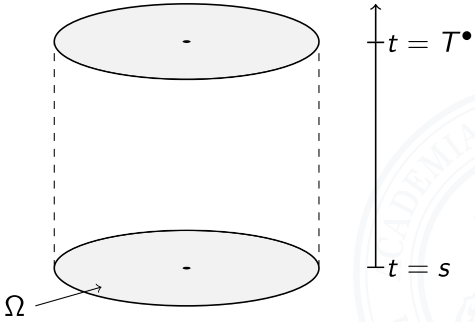

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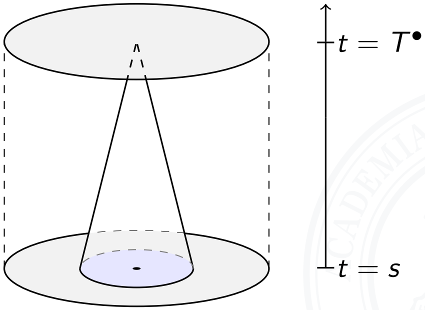
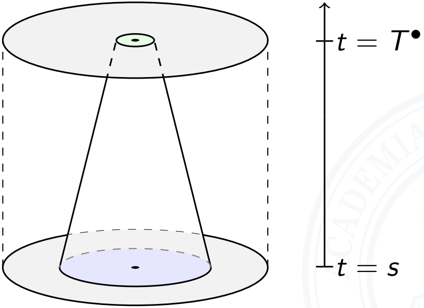


Proof idea/outline

time-space localize to **backwards light cone** from $(T^\bullet, \mathrm{x}_0)$:

$$\Lambda(s, T^\bullet; \delta) \coloneqq \Big\{(t, \mathrm{x}) \in [s, T^\bullet] \times \overline{\Omega} \colon |\mathrm{x} - \mathrm{x}_0| < T^\bullet - t + \delta\Big\}$$



Proof idea/outline

- **bootstrapping**: $(\mathrm{x}_0 \in \overline{\Omega})$

$$\lim_{t \nearrow T^\bullet} \|y\|_{\mathrm{L}^\infty_t \mathrm{L}^6_\mathrm{x}(\Lambda(t,T^\bullet;0))} = 0$$

$$\Downarrow$$

$$\exists\, t_0(\mathrm{x}_0)\colon \quad \|y\|_{\mathrm{L}^4_t \mathrm{L}^{12}_\mathrm{x}(\Lambda(t_0(\mathrm{x}_0),T^\bullet;0))} < \infty$$

$$\Downarrow$$

$$\exists\, \alpha(\mathrm{x}_0) > 0\colon \quad \lim_{t \nearrow T^\bullet} \|y\|_{\mathrm{L}^\infty_t \mathrm{L}^6_\mathrm{x}(\Lambda(t,T^\bullet;\alpha(\mathrm{x}_0)))} = 0$$

$$\Downarrow$$

$$\exists\, t_0(\mathrm{x}_0), \alpha(\mathrm{x}_0) > 0\colon \quad \|y\|_{\mathrm{L}^4_t \mathrm{L}^{12}_\mathrm{x}(\Lambda(t_0(\mathrm{x}_0),T^\bullet;\alpha(\mathrm{x}_0)))} < \infty$$

- cover/compactness of $\overline{\Omega}$: $\longrightarrow$ $\exists\, t^*\colon \|y\|_{\mathrm{L}^4(t^*,T^\bullet;\mathrm{L}^{12}(\Omega))} < \infty$

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Optimal control

$$\min_{(y,u)} \quad \ell(y,u)$$
$$\text{s.t.} \quad y'' - \Delta y + y^5 = u \qquad \text{on } (0,T)\times\Omega,$$
$$\|u(t)\|_2 \le \omega(t) \quad \text{on } (0,T)$$

- objective function

$$\ell(y,u) = \frac{1}{2}\|y(T) - y_d\|_2^2 + \frac{\gamma}{4}\|y\|^4_{L^4(0,T;L^{12}(\Omega))} + \beta_1\|u\|_{L^1(0,T;L^2(\Omega))} + \frac{\beta_2}{2}\|u\|^2_{L^2(0,T;L^2(\Omega))}$$

- consider $0 \le \omega \in L^1(0,T)$, allow: $\omega(t) \to 0$ as $t \to \overline{t} \in (0,T)$

Existence of optimal controls

- $\omega \in \mathrm{L}^1(0,\mathrm{T})$ $\longrightarrow$ $\mathcal{U}_{\mathrm{ad}} \subset \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^2(\Omega))$ weakly compact
- mild$^+$ solution $\sim$ weak solution $\longrightarrow$ proof quite standard

Theorem

There exists a global solution $(\bar{y},\bar{u})$ to the OC problem for the CWE with $\bar{y}[\cdot] \in \mathrm{L}^\infty(0,\mathrm{T};\mathcal{E})$ and $\bar{u} \in \mathcal{U}_{ad}$.

- *If $\gamma = 0$, then $\bar{y}$ is a (possibly non-unique) weak solution to the CWE with data $\bar{u}$.*
- *If $\gamma > 0$, then $\bar{y} \in \mathrm{L}^4(0,\mathrm{T};\mathrm{L}^{12}(\Omega))$ is the unique mild$^+$ solution to the CWE with data $\bar{u}$.*

- keep $\gamma > 0$ from now on

Control-to-state operator

- control-to-state operator $\mathcal{S}\colon u \mapsto y_u$ well defined

$$\mathcal{S}\colon \mathrm{L}^1(0,\mathsf{T};\mathrm{L}^2(\Omega)) \\ \longrightarrow \quad \Big\{y\colon y[\cdot] \in \mathrm{C}\big([0,\mathsf{T}];\mathcal{E}\big)\Big\} \cap \mathrm{L}^4(0,\mathsf{T};\mathrm{L}^{12}(\Omega)) =: \mathcal{Y}$$

  and (twice) continuously differentiable,

- derivative $\mathcal{S}'(u)h$ given by mild$^+$ solution z_h to linear WE

$$z'' - \Delta z + 5y_u^4 z = h, \quad z[0] = 0$$

- 2nd derivative $\mathcal{S}''(u)(h_1, h_2)$ given by mild$^+$ solution $z_{h_1 h_2}$ to

$$z'' - \Delta z + 5y_u^4 z + 20 y_u^3 z_{h_1} z_{h_2} = 0, \quad z[0] = 0.$$

Optimality conditions

Theorem

Let $\bar{u}$ be a locally optimal control to the OC problem for the CWE. Then there exists $\bar{\lambda} \in \partial\|\cdot\|_{L^1(0,T;L^2(\Omega))}(\bar{u}) \subset L^\infty(0,T;L^2(\Omega))$ such that

$$\int_0^T \left(\bar{p}(t) + \beta_1\bar{\lambda}(t) + \beta_2\bar{u}(t), v(t)\right)_\Omega \mathrm{d}t \geq 0 \quad \textit{for all } v \in \mathcal{T}(\bar{u}),$$

where $\bar{p}$ is the adjoint state: $\left(\Psi(y) := \frac{1}{4}\|y\|^4_{L^4(0,T;L^{12}(\Omega))}\right)$

$$\bar{p} = \mathcal{S}(\bar{u})^*\left[\delta_T^*(y_{\bar{u}}(T) - y_d) + \gamma\Psi''(y_{\bar{u}})\right] \in L^\infty(0,T;L^2(\Omega)).$$

- structure: $\bar{u} \in L^\infty(0,T;L^2(\Omega))$ if $\beta_2 > 0$ & **sparsity** [CHW16]

$$\|\bar{p}(t)\|_2 < \beta_1 \implies \bar{u}(t) = 0 \implies \|\bar{p}(t)\|_2 \leq \beta_1$$

Lagrange multiplier

$$L(u,\mu) := \ell(y_u,u) + \int_0^T \mu(t)\big(\|\bar u(t)\|_2 - \omega(t)\big)\,\mathrm{d}t$$

Theorem

Let $\bar u$ be locally optimal for the OC problem for the CWE. Then exists unique ***Lagrange multiplier*** $\bar\mu \in \mathrm{L}^\infty(\mathcal{A}_+ \cup \mathcal{I})$ *such that*

$$\bar p(t) + \beta_1 \bar\lambda(t) + \beta_2 \bar u(t) + \bar\mu(t)\frac{\bar u(t)}{\|\bar u(t)\|_2} = 0 \quad \text{a.e. on } \mathcal{A}_+ \cup \mathcal{I}$$

with $\bar\mu \geq 0$ *and* $\bar\mu(t)\big(\|\bar u(t)\|_2 - \omega(t)\big) = 0$.

- $\mathcal{A}_+ := [\,\|\bar u\|_2 = \omega > 0\,]$, $\quad \mathcal{I} := [\,\|\bar u\|_2 < \omega\,]$
- proof "by hand": Hahn-Banach theorem in $\mathrm{L}^1(\mathcal{A}_+)$ (from VI)
- $\bar\mu \in \mathrm{L}^\infty(\mathcal{A}_+ \cup \mathcal{I})$ necessary: take $\|\cdot\|_2$ norms

Lagrange multiplier

$$L(u,\mu) := \ell(y_u,u) + \int_0^T \mu(t)\big(\|\bar u(t)\|_2 - \omega(t)\big)\,\mathrm{d}t$$

Theorem

Let $\bar u$ be locally optimal for the OC problem for the CWE. Then exists unique ***Lagrange multiplier*** $\bar\mu \in \mathrm{L}^\infty(\mathcal{A}_+ \cup \mathcal{I})$ *such that*

$$\bar p(t) + \beta_1\bar\lambda(t) + \beta_2\bar u(t) + \bar\mu(t)\frac{\bar u(t)}{\|\bar u(t)\|_2} = 0 \quad \textit{a.e. on } \mathcal{A}_+ \cup \mathcal{I}$$

with $\bar\mu \geq 0$ *and* $\bar\mu(t)\big(\|\bar u(t)\|_2 - \omega(t)\big) = 0$.

- $\mathcal{A}_+ := [\,\|\bar u\|_2 = \omega > 0\,], \quad \mathcal{I} := [\,\|\bar u\|_2 < \omega\,]$
- proof "by hand": Hahn-Banach theorem in $\mathrm{L}^1(\mathcal{A}_+)$ (from VI)
- $\bar\mu \in \mathrm{L}^\infty(\mathcal{A}_+ \cup \mathcal{I})$ necessary: take $\|\cdot\|_2$ norms

Second order conditions

Theorem (Necessary condition)

Let $\bar{u} \in \mathcal{U}_{ad}$ be locally optimal for the OCP of the CWE. Then

$$\partial^2_{uu} L(\bar{u}, \bar{\mu}; v^2) \geq 0 \quad \textit{for all } v \in C(\bar{u}) := \Big\{ v \in \mathcal{T}(\bar{u}) \colon \ell_r'(\bar{u}; v)) = 0 \Big\}.$$

Theorem (Sufficient condition)

Let $\beta_2 > 0$. Assume that $\bar{u} \in \mathcal{U}_{ad}$ satisfies

$$\ell_r''(\bar{u}; v^2) > 0 \quad \textit{for all } v \in C(\bar{u}) \setminus \{0\}.$$

Then $\bar{u}$ is a local minimum: exist $\varepsilon, \delta > 0$ such that for all $u \in \mathcal{U}_{ad}$:

$$\ell_r(\bar{u}) + \frac{\delta}{2} \|u - \bar{u}\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega))} \leq \ell_r(u) \quad \textit{if} \quad \|u - \bar{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega))} < \varepsilon.$$

Wrap-up

- optimal control of a defocusing CWE
- unique well posed global solutions with additional integrability
- existence of optimal controls, sparsity-in-time control
- FONC, SONC, SOSC with $\beta_2 > 0$

Read it up: [KM20] (JMPA 2020).

open:

- SOSC with $\beta_2 = 0$?
- control explosive (defocusing!) equations by tracking at T? ([AQ05])

Wrap-up

- optimal control of a defocusing CWE
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open:

- SOSC with $\beta_2 = 0$?
- control explosive (defocusing!) equations by tracking at T? ([AQ05])

Outlook: Boundary control?

$$\min_{(y,u)} \quad \ell(y,u)$$
$$\text{s.t.} \quad y'' - \Delta y + y^5 = 0 \quad \text{on } (0,\mathrm{T}) \times \Omega,$$
$$\partial_\nu y = u \quad \text{on } (0,\mathrm{T}) \times \partial\Omega$$

- Decompose:

$$\left.\begin{aligned} v'' - \Delta v &= 0, \\ \partial_\nu v &= u \end{aligned}\right\} \text{(LI)}, \qquad \left.\begin{aligned} w'' - \Delta w + (w+v)^p &= 0, \\ \partial_\nu w &= 0 \end{aligned}\right\} \text{(NLH)}$$

- Regularity transfer $u \mapsto v$ in (LI)? **Boundary Strichartz estimate**?

$$\|v\|_{\mathrm{L}^r(0,\mathrm{T};\mathrm{L}^s(\Omega))} \lesssim \|u\|_X \qquad (\text{e.g. } X = \mathrm{L}^\alpha(0,\mathrm{T};\mathrm{L}^\beta(\partial\Omega)))$$

determines w and/or p in (NLH)!

Outlook: Boundary control?

$$\left.\begin{aligned} v'' - \Delta v &= 0, \\ \partial_\nu v &= u \end{aligned}\right\} \quad \xleftrightarrow[\text{Duality}]{\text{Transposition}} \quad \begin{cases} z'' - \Delta z = f, \\ \partial_\nu z = 0 \end{cases}$$

- Interpolate energy estimate + Strichartz for z & transpose:

$$\|v\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \lesssim \|u\|_{\mathrm{L}^{\frac{4}{4-\theta}}(0,T;\mathrm{L}^{\frac{8}{\theta+6}}(\partial\Omega))}$$

- Make-a-wish, **indirect**:

$$\|z\|_{X(0,T;\partial\Omega)} \lesssim \|f\|_{\mathrm{L}^{\frac{5}{4}}(0,T;\mathrm{L}^{\frac{10}{9}}(\Omega))} \qquad \textbf{?}$$

or **direct**:

$$\|v\|_{\mathrm{L}^5(0,T;\mathrm{L}^{10}(\Omega))} \lesssim \|u\|_{X'(0,T;\partial\Omega)} \qquad \textbf{?}$$

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Second order considerations

- $j(\cdot) = \|\cdot\|_{L^1(0,T;L^2(\Omega))} \qquad \longrightarrow \qquad$ directional derivatives $j'(\bar{u};\cdot)$ exist
- critical cone:

$$C(\bar{u}) := \Big\{ v \in \mathcal{T}(\bar{u})\colon\ F'(\bar{u})v + \beta_1 j'(\bar{u}; v) = 0 \Big\}$$

- nice: F twice cont. differentiable $\qquad (\Psi(u) = \frac{1}{4}\|y_u\|^4_{L^4(0,T;L^{12}(\Omega))})$

$$F''(\bar{u})h^2 = \|z_h(T)\|_2^2 - \int_0^T \big(\bar{p}(t), 20y_{\bar{u}}^3(t)z_h^2(t)\big)_\Omega \,\mathrm{d}t + \gamma\Psi''(\bar{u})z_h^2 + \beta_2\|h\|^2_{L^2(0,T;L^2(\Omega))}$$

and $h \mapsto F''(\bar{u})h^2$ weakly lower semicontinuous (Vitali)

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Second order considerations

- not so nice: $j''(\bar{u}; v^2)$ (and multiplier term) may be infinite:

$$\int_{\left[\|\bar{u}\|_2 \neq 0\right]} \|\bar{u}(t)\|_2^{-1} \left[\|v(t)\|_2^2 - \left(\frac{\left(\bar{u}(t), v(t)\right)_\Omega}{\|\bar{u}(t)\|_2} \right)^2 \right] \mathrm{d}t$$

$\longrightarrow$ problem with second order Taylor expansions for j

$\longrightarrow$ rely on directions v_k with $v_k = 0$ on $\left[\|\bar{u}\|_2 < 1/k\right]$

- approximate general direction v by v_k, monotone convergence:

$$0 \leq j''(\bar{u}; v_k^2) \nearrow j''(\bar{u}; v^2) \quad \text{as } k \to \infty$$

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Second order necessary conditions

Theorem

Let $\bar{u} \in \mathcal{U}_{ad}$ be locally optimal for the OCP of the CWE. Then

$$L''_{uu}(\bar{u}, \bar{\mu}; v^2) \geq 0 \quad \textit{for all } v \in C(\bar{u}).$$

- Taylor expand, use FONC, ... ([CHW16])

$$0 \leq \ell(y_{\bar{u}+\rho v}, \bar{u} + \rho v) - \ell(y_{\bar{u}}, \bar{u}) = \dots$$

- **but** constraint curvature $\longrightarrow$ $\bar{u} + \rho v \notin \mathcal{U}_{\mathrm{ad}}$ in general
- construct continuous nonlinear path $\rho \mapsto u_\rho \in \mathcal{U}_{\mathrm{ad}}$ with

$$\lim_{\rho \searrow 0} u_\rho = \bar{u}, \quad \frac{\mathrm{d}}{\mathrm{d}\rho} u_\rho(0) = v$$

- second stage approx. $v_k \to v$ $\longrightarrow$ Taylor for j ($v_k \notin C(\bar{u})$!)

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Second order sufficient conditions

Theorem

Let $\beta_2 > 0$. Assume that $\bar{u} \in \mathcal{U}_{ad}$ satisfies

$$\ell_r''(\bar{u}; v^2) > 0 \quad \textit{for all } v \in C(\bar{u}) \setminus \{0\}.$$

Then $\bar{u}$ is a local minimum: exist $\varepsilon, \delta > 0$ such that

$$\ell_r(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega))} \leq \ell_r(u) \qquad \textit{for all } u \in \mathcal{U}_{ad} \textit{ with } \|u - \bar{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega))} < \varepsilon.$$

- standard proof by contradiction (modulo cutoff/Taylor for j)
- good $\mathrm{L}^2(0, \mathrm{T}; \mathrm{L}^2(\Omega))$ optimality $\longrightarrow$ stability results

A neat thing

$$\int_0^T \int_\Omega \bar{p}(t) y_{\bar{u}}^3(t) z_k^2(t) \,\mathrm{d}x\,\mathrm{d}t \quad \text{for} \quad h_k \rightharpoonup h \text{ in } \mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$$

- obtain: $S'(\bar{u})h_k = z_k \rightharpoonup z = S'(\bar{u})h$ in $\mathcal{Y}$
- z_k bounded in $\mathrm{L}^5(0,T;\mathrm{L}^{10}(\Omega))$
- compactness: $S'(\bar{u})h_{k_\ell} = z_{k_\ell} \to z = S'(\bar{u})h$ a.e. on $(0,T) \times \Omega$

A neat thing

$$\int_0^T \int_\Omega \bar{p}(t) y_{\bar{u}}^3(t) z_k^2(t) \, \mathrm{d}x \, \mathrm{d}t \quad \text{for} \quad h_k \rightharpoonup h \text{ in } \mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$$

$$\leq \|p\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \|y_{\bar{u}}\|^3_{\mathrm{L}^5(0,T;\mathrm{L}^{10}(\Omega))} \|z_k\|^2_{\mathrm{L}^5(0,T;\mathrm{L}^{10}(\Omega))}$$

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no integrable majorant!

A neat thing

$$\int_{E_t}\int_{E_\mathrm{x}} \bar{p}(t) y_{\bar{u}}^3(t) z_k^2(t)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for} \quad h_k \rightharpoonup h \text{ in } \mathrm{L}^1(0,\mathsf{T};\mathrm{L}^2(\Omega))$$

$$\leq \|p\|_{\mathrm{L}^\infty(0,\mathsf{T};\mathrm{L}^2(\Omega))} \|y_{\bar{u}}\|^3_{\mathrm{L}^5(E_t;\mathrm{L}^{10}(E_\mathrm{x}))} \|z_k\|^2_{\mathrm{L}^5(0,\mathsf{T};\mathrm{L}^{10}(\Omega))}$$

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no integrable majorant!

A neat thing

$$\int_{E_t}\int_{E_\mathrm{x}} \bar{p}(t) y_{\bar{u}}^3(t) z_k^2(t)\,\mathrm{d}\mathrm{x}\,\mathrm{d}t \quad \text{for} \quad h_k \rightharpoonup h \text{ in } \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^2(\Omega))$$

$$\lesssim \|y_{\bar{u}}\|^3_{\mathrm{L}^5(E_t;\mathrm{L}^{10}(E_\mathrm{x}))} \quad \xrightarrow{|E_t \times E_x| \to 0} \quad 0 \quad \text{uniformly in } k$$

- obtain: $S'(\bar{u})h_k = z_k \rightharpoonup z = S'(\bar{u})h$ in $\mathcal{Y}$
- z_k bounded in $\mathrm{L}^5(0,\mathrm{T};\mathrm{L}^{10}(\Omega))$
- compactness: $S'(\bar{u})h_{k_\ell} = z_{k_\ell} \to z = S'(\bar{u})h$ a.e. on $(0,\mathrm{T}) \times \Omega$

no integrable majorant . . . but uniform integrability!

$\longrightarrow$ **Vitali** convergence theorem allows limit passage as $h_k \rightharpoonup h$