Inverse design for the chromatography system

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(C)
$$\begin{cases} \partial_t u_1 + \partial_x \left(\dfrac{u_1}{1+u_1+u_2} \right) = 0, & t>0,\ x\in\mathbb{R}, \\ \partial_t u_2 + \partial_x \left(\dfrac{u_2}{1+u_1+u_2} \right) = 0, & t>0,\ x\in\mathbb{R}, \\ u_1(0,x) = \bar{u}_1(x), & x\in\mathbb{R}, \\ u_2(0,x) = \bar{u}_2(x), & x\in\mathbb{R}, \end{cases}$$

where $u_1, u_2 \in \mathbb{R}_+$ are the components' concentrations.

[Bianchini, *SIMA* 2001]: For any initial condition $\bar{u}_1, \bar{u}_2 \in L^\infty(\mathbb{R})$, the Cauchy problem (C) admits a unique *entropy admissible solution*.

In particular, the system admits a *continuous semi-group of solutions*

$$\begin{aligned} S^+ : \quad & \mathbb{R}_+ \times L^\infty(\mathbb{R};\mathbb{R}^2) && \to && L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2) \\ & (t, U_0 = (\bar{u}_1, \bar{u}_2)) && \mapsto && S_t^+(U_0). \end{aligned}$$

Changing variables

Using the change of variables

$$v := u_1 + u_2 \quad \text{and} \quad w := u_1 - u_2,$$

the system (C) reduces to the coupling between a scalar conservation law and a linear continuity equation:

$$\text{(T)} \qquad \begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t w + \partial_x \left(\dfrac{w}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = \bar{u}_1(x) + \bar{u}_2(x), & x \in \mathbb{R}, \\ w(0,x) = \bar{u}_1(x) - \bar{u}_2(x), & x \in \mathbb{R}. \end{cases}$$

Triangular system: scalar conservation law + transport equation with OSL velocity.

[Panov, *Springer* 2008]: Let us consider the (more general) problem

(P)
$$\begin{cases} \partial_t(A\rho) + \partial_x(B\rho) = 0, & t > 0,\ x \in \mathbb{R}, \\ A(0,x)\rho(0,x) = A(0,x)\rho_0(x), & x \in \mathbb{R}, \end{cases}$$

under the assumptions

1. A and B in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$;
2. $\partial_t A + \partial_x B = 0$ in $\mathcal{D}'((0,+\infty) \times \mathbb{R})$;
3. there exists $N : \mathbb{R} \to \mathbb{R}$ such that $\varepsilon N(\varepsilon) \to 0$ as ε tends to zero and for all $\varepsilon > 0$, $|B| \leq N(\varepsilon)(A + \varepsilon)$ a.e. in $(0,+\infty) \times \mathbb{R}$;
4. $\operatorname{ess-}\lim\limits_{t \to 0^+} A(t,\cdot) = A(0,\cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $A(0,\cdot) \in L^\infty(\mathbb{R})$.

For any given bounded initial condition ρ_0, there exists a bounded function ρ, called *generalized solution* of (P), such that, for any test function φ in $C_0^\infty([0,+\infty) \times \mathbb{R})$,

$$\int_0^{+\infty} \int_{\mathbb{R}} \Big((A\rho)\partial_t\varphi + (B\rho)\partial_x\varphi \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} A(0,x)\rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0.$$

Moreover, every generalized solution ρ enjoys the following properties.

Strong traces The initial condition is satisfied in the sense that in $L^1_{\mathrm{loc}}(\mathbb{R})$

$$\operatorname*{ess\text{-}lim}_{t\to 0^+} A(t,x)\rho(t,x) = A(0,x)\rho_0(x),$$

and for all $T>0$, there exists $\operatorname*{ess\text{-}lim}_{t\to T^-} A(t,x)\rho(t,x)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Reversibility If ρ is a generalized solution of problem (P) and the identity $A(T,x)\rho(T,x) = A(T,x)\rho_T(x)$ holds in the sense of strong traces, then $t \mapsto \rho(T-t)$ is a generalized solution of

(P1)
$$\begin{cases} \partial_t(A\rho) - \partial_x(B\rho) = 0, & t>0,\ x\in\mathbb{R}, \\ A(0,x)\rho(0,x) = A(0,x)\rho_T(x), & x\in\mathbb{R}. \end{cases}$$

Uniqueness If $A(0,x)\rho_0(x) = 0$ a.e. on $\mathbb{R}$ then $A(t,x)\rho(t,x) = 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}$.

Renormalization for any function μ in $C(\mathbb{R})$ the function $\mu \circ \rho$ satisfies

(P2)
$$\begin{cases} \partial_t\left(A(\mu(\rho))\right) + \partial_x\left(B(\mu(\rho))\right) = 0, & t>0,\ x\in\mathbb{R}, \\ \left(\mu(\rho)\right)(0,x) = \mu\left(\rho_0(x)\right), & x\in\mathbb{R}, \end{cases}$$

in the sense of distributions.

Well-posedness of the Chromatography system

In the case of system (T), the scalar equation

(CS) $$\partial_t v + \partial_x \left(\frac{v}{1+v} \right) = 0, \qquad t > 0,\ x \in \mathbb{R},$$

admits a unique entropy solution in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ starting from any initial condition $v_0 \in L^\infty(\mathbb{R})$.

This allows us to define the divergence-free vector field $t \mapsto (A(t,x), B(t,x))$, for $A(t,x) = v(t,x)$ and $B(t,x) = \dfrac{v(t,x)}{1+v(t,x)}$, satisfying all of the hypothesis above.

Then Panov's theorem guarantees that, for any given $z_0 \in L^\infty(\mathbb{R})$, there exists a unique generalized solution in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ of

(TP) $$\begin{cases} \partial_t (vz) + \partial_x \left(\dfrac{vz}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ z(0,x) = z_0(x), & x \in \mathbb{R}. \end{cases}$$

From this, we can recover the solution for the chromatography system in the form (T) and (C):

- the solution w of the transport equation appearing in (T) can be seen as $w = vz$ if we set $z_0 = (\bar{u}_1 - \bar{u}_2)/v_0$;
- the solution (u_1, u_2) of (C) can be seen as $(u_1 = vz_1, u_2 = vz_2)$ if z_i (for $i \in \{1, 2\}$) is the solution of (TP) corresponding to the initial condition $z_{0,i} = \bar{u}_i/v_0$.

In both cases, since the z satisfies the maximum principle and we consider $v_0 = \bar{u}_1 + \bar{u}_2$ with $\bar{u}_i \geq 0$, we have that $\|z\|_{L^\infty(\mathbb{R})} \leq 1$.

Definition 1 (Solution of the chromatography system in the form (C))

Let $\bar{U} = (\bar{u}_1, \bar{u}_2) \in L^\infty(\mathbb{R}; \mathbb{R}^2)$ be the initial conditions imposed to the system (C). A function $U = (u_1, u_2) \in L^\infty((0,T) \times \mathbb{R}; \mathbb{R}^2)$ is a *strong generalized entropy solution* for the system (C) if $U = \left(\frac{v+w}{2}, \frac{v-w}{2}\right)$, where

- the function v is the Kružkov entropy solution of

(CS)
$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v}\right) = 0, & t \in (0,T),\ x \in \mathbb{R}, \\ v(0,x) = \bar{u}_1(x) + \bar{u}_2(x), & x \in \mathbb{R} :=; \end{cases}$$

- the function w is given by $w = vz$, where z is the solution of (TP) in the weak sense with initial datum $z_0(x) = \dfrac{\bar{u}_1(x) - \bar{u}_2(x)}{\bar{u}_1(x) - \bar{u}_2(x)}$ and coefficients $A = v$ and $B = \dfrac{v}{1+v}$.

In the above definition, the value of $\dfrac{\bar{u}_1(x) - \bar{u}_2(x)}{\bar{u}_1(x) - \bar{u}_2(x)}$ can be taken arbitrary ± 1 at points where $\bar{U} = 0$.

Definition 2 (Renormalized entropy solution of the chromatography system in the form (C))

The function $U \in L^\infty((0,T) \times \mathbb{R}; \mathbb{R}^2)$ is a renormalized entropy solution for the system (T) if

- the function v is the Kružkov entropy solution of (CS);
- the function w is given by $w = zv$ where, for any test function $\varphi \in C_0^\infty([0,T) \times \mathbb{R})$ and any continuous function μ, z satisfies

$$\int_0^T \int_{\mathbb{R}} \left(v\mu(z)\partial_t\varphi + \frac{1}{1+v}\mu(w)\partial_x\varphi \right) \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} v(0,x)\mu(z_0(x))\varphi(0,x)\,\mathrm{d}x = 0.$$

Entropy/entropy-flux pairs

The entropy/entropy-flux pairs for the systems (T) take the following form:

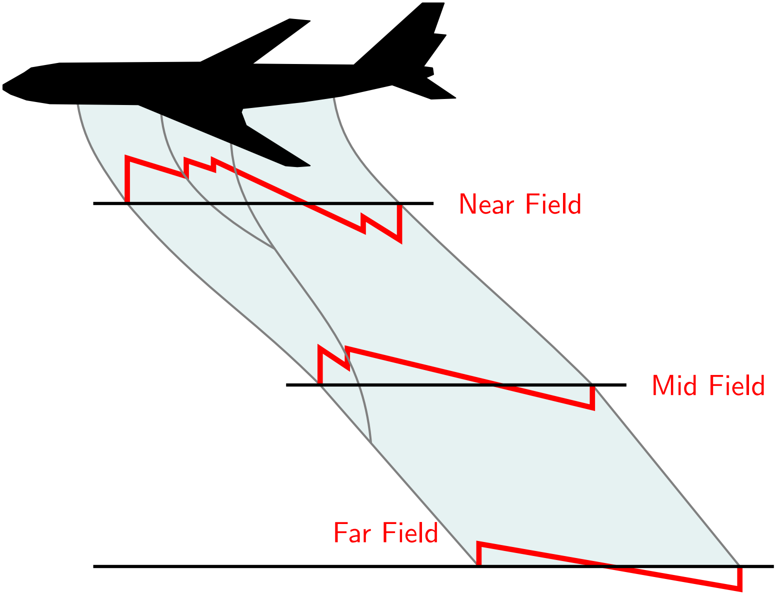
$$\mathcal{E}(v,w) := \eta(v) + v\mu\left(\frac{w}{v}\right),$$
$$\mathcal{Q}(v,w) := q(v) + \frac{v}{1+v}\mu\left(\frac{w}{v}\right),$$

where η is any entropy function for (CS) and $\mu \in C(\mathbb{R})$. Therefore, the strong generalized entropy solutions coincide with the entropy solutions for the system.

Inverse design problem

1. for any given $T > 0$, characterize the set of profiles $U_T = (u_1^T, u_2^T) \in L^\infty(\mathbb{R}; \mathbb{R}^2)$ for which there exists at least one initial condition $(\bar{u}_1, \bar{u}_2) \in L^\infty(\mathbb{R}; \mathbb{R}^2)$ such that $S_T^+(\bar{u}_1, \bar{u}_2) = (u_1^T, u_2^T)$;
2. for each of such attainable profiles, characterize the set of initial data leading to them, $\mathfrak{I}(U_T)$;
3. for profiles that cannot be attained by a trajectory of the system, recover the initial data leading to their "best possible approximation".

Sonic-boom minimization and inverse design for the Burgers' equation



Scalar problem: attainable profiles

The set of states in $L^\infty(\mathbb{R})$ attainable by Kružkov entropy solutions at time $T > 0$ to scalar conservation laws with strictly convex flux was characterized in [Ancona–Marson, *SICON* 1998] as follows:

$$\text{(AS)} \qquad \mathcal{A}_T(\mathbb{R}, f) = \Big\{ u \in L^\infty(\mathbb{R}) : \ \exists \rho : \mathbb{R} \to \mathbb{R}, \text{ right continuous, non-decreasing such that } f'(u) = \frac{x - \rho(x)}{T} \Big\}$$

Oleinik's condition

Furthermore, for every $u_T \in \mathcal{A}_T(\mathbb{R}, f)$, there exists a unique *isentropic solution* $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ that verifies $u(T, \cdot) = u_T$.

Scalar problem: backward reconstruction

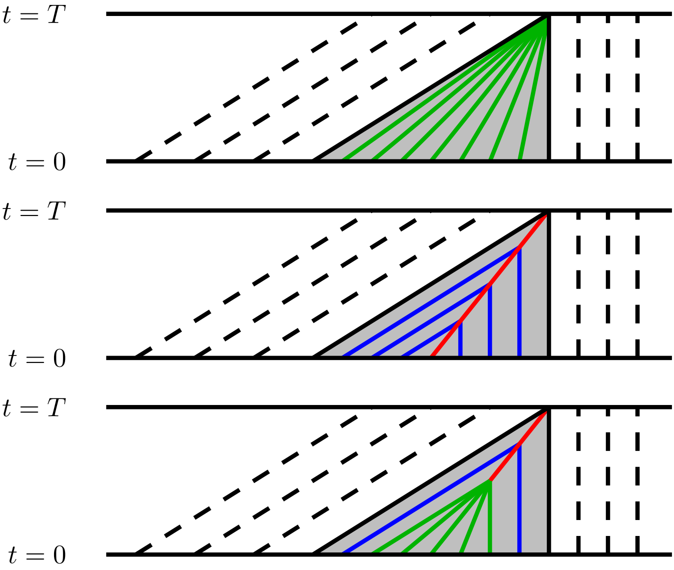


Figure: Multiple initial data may lead to the same target profile.

[Colombo–Perrollaz, *JMPA* 2020] & [Liard–Zuazua, *IEEE* 2021] characterized the set of inverse designs as follows.

Theorem 3 (Inverse design for scalar conservation laws)

Let us consider a strictly convex scalar conservation law, fix $T > 0$*, and let* $v_T \in \mathcal{A}_T(\mathbb{R}, f)$*. Then, the initial data* $\tilde{u}_0 \in L^\infty(\mathbb{R})$ *verifies* $\mathcal{S}_T^+(\tilde{u}_0) = u_T$ *if and only if the following statement holds:*

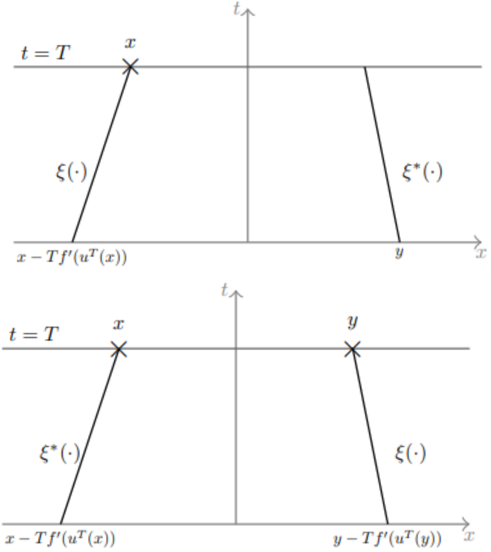
1) *for any* $(x, y) \in X(u_T) \times \mathbb{R}$

$$\int_{x - Tf'(u_T(x))}^{y} \mathcal{S}_T^-(u_T)(s)\,\mathrm{d}s \leq \int_{x - Tf'(u_T(x))}^{y} \tilde{u}_0(s)\,\mathrm{d}s, \tag{1}$$

2) *for any* $(x, y) \in X(u_T)^2$

$$\int_{x - Tf'(u_T(x))}^{y - Tf'(u_T(y))} \mathcal{S}_T^-(u_T)(s)\,\mathrm{d}s = \int_{x - Tf'(u_T(x))}^{y - Tf'(u_T(y))} \tilde{u}_0(s)\,\mathrm{d}s, \tag{2}$$

where $X(u_T)$ *is the set of points of approximate continuity of* u_T*.*


t
t = T
x
ξ(·)
ξ*(·)
x − Tf′(u^T(x))
y
x
t
t = T
x
y
ξ*(·)
ξ(·)
x − Tf′(u^T(x))
y − Tf′(u^T(y))
x

Attainable profiles for the Chromatography system

[Andreianov–Donadello–Ghoshal–Shyam–Razafison, *J. Evol. Eq.* 2015]: The set of states in $L^\infty(\mathbb{R})$ that are attainable by entropy solutions of system (T) is given by

(AT) $$\mathfrak{A}_T(\mathbb{R}) = \mathcal{A}_T(\mathbb{R}, f) \times L^\infty(\mathbb{R}).$$

NB: A generic element of the set of the attainable profiles for (T) at time $T > 0$,

$$\mathfrak{A}_T(\mathbb{R}) = \left\{(v_T, w_T) : v_T \in \mathcal{A}_T\left(\mathbb{R}, v \mapsto \frac{v}{1+v}\right) \text{ and } w_T \in L^\infty(\mathbb{R})\right\},$$

does not correspond to an attainable profile for the system (C) because it might happen that $((v_T + w_T)(x), (v_T - w_T)(x))$ is not in $\mathbb{R}^2_+$ for almost every $x \in \mathbb{R}$, while all physically relevant solutions of (C) are in $L^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}^2_+)$.

Therefore, we define

$$\mathsf{A}_T(\mathbb{R}) = \Big\{(v_T, w_T) : v_T \in \mathcal{A}_T\left(\mathbb{R}, v \mapsto \frac{v}{1+v}\right) \text{ and there exists } z \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [-1,1]) \text{ such that } w_T = z v_T\Big\}$$

and we say that $U_T = (u_1^T, u_2^T)$ is attainable for system (C) if and only if $(v_T := u_1^T + u_2^T, w_T = u_1^T - u_2^T) \in \mathsf{A}_T$.

Backward reconstruction for the Chromatography system

Theorem 4 (Characterization and properties of the set of inverse designs)

1) *Given* $U_T = (u_1^T, u_2^T) \in L^\infty(\mathbb{R}; \mathbb{R}^2_+)$ *such that* $V_T = (v_T = u_1^T + u_2^T, w_T = u_1^T - u_2^T) \in \mathsf{A}_T(\mathbb{R})$*, the set of inverse design*

$$\mathfrak{i}(U_T) = \{U_0 = (u_1^0, u_2^0) \in L^\infty(\mathbb{R}; \mathbb{R}_+) \times L^\infty(\mathbb{R}; \mathbb{R}_+) : S_T^+(U_0) = U_T\}$$

can be characterized as follows:

$$\mathfrak{i}(U_T) = \left\{ \begin{aligned} U_0 = (u_1^0, u_2^0) \in L^\infty(\mathbb{R}; \mathbb{R}_+) \times L^\infty(\mathbb{R}; \mathbb{R}_+) : \\ u_1^0 + u_2^0 \in \mathcal{I}(v_T) \text{ and } u_i^0 = v_0 \mathscr{S}_T^-[\mathcal{S}_*(v_0)](u_i^T/v^T), \coloneqq i = 1,\, 2 \end{aligned} \right\}.$$

2) *Given* $V_T = (v_T, w_T) \in \mathfrak{A}_T(\mathbb{R})$*, the set of inverse design*

$$\mathfrak{I}(V_T) = \{V_0 \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) : \mathfrak{S}_T^+(V_0) = V_T\}$$

can be characterized as follows:

$$\mathfrak{I}(V_T) = \{(v_0, w_0) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) :\ v_0 \in \mathcal{I}(v_T) \text{ and } w_0 = \mathscr{S}_T^-[\mathcal{S}_*(v_0)](w_T)\}.$$

In both of the above cases, the 1–to–1 correspondence between the elements of $\mathcal{I}(v_T)$ and the elements of $\mathfrak{i}(U_T)$ and $\mathfrak{I}(V_T)$, together with the results on $\mathcal{I}(v_T)$ proved in [Colombo–Perrollaz, *JMPA* 2020] yield the following properties:

(T1) the set $\mathfrak{I}(V_T)$ is closed with respect to the $L^1_{\mathrm{loc}} \times L^1_{\mathrm{loc}}$ topology;

(T2) the set $\mathfrak{I}(V_T)$ has empty interior with respect to the $L^1_{\mathrm{loc}} \times L^1_{\mathrm{loc}}$ topology;

(G1) the sets $\mathfrak{I}(V_T)$ and $\mathfrak{i}(U_T)$ reduce to a singleton if and only if $v_T \in C(\mathbb{R})$.

Let us describe the numerical method:

1. first, we use the entropic solver to get the isentropic solution of the first equation;
2. then, the numerical backward resolution of the transport equation is based on considering the auxiliary forward problem

$$\partial_t w + \partial_x \left(-g(v)w\right) = 0, \qquad t > 0,\ x \in \mathbb{R},$$

where v is an entropic solution of the first equation.

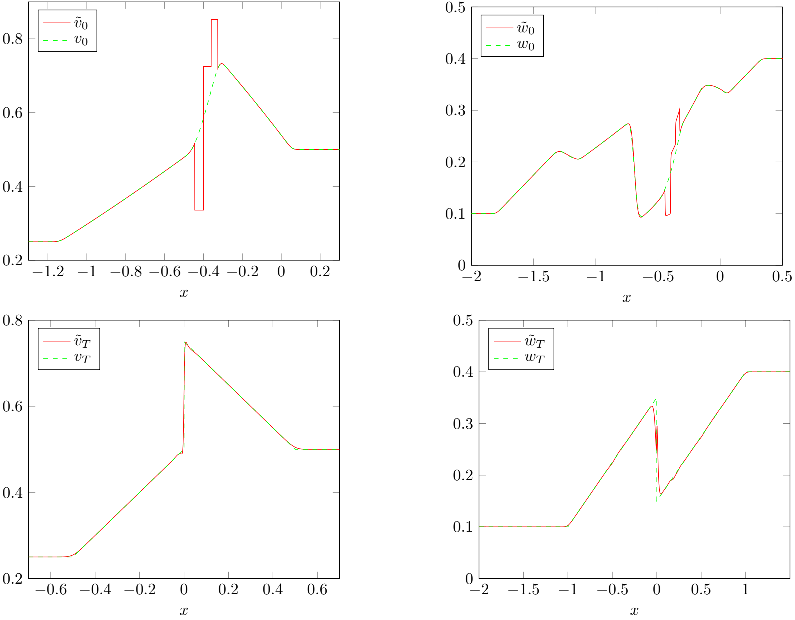
We consider

$$v_T(x) = \begin{cases} 0.25, & \text{if } x < -0.5, \\ 0.25 + 0.5(x + 0.5), & \text{if } -0.5 \leq x < 0, \\ 0.75 - 0.5, & \text{if } 0 \leq x < 0.5, \\ 0.5, & \text{if } 0.5 \leq x, \end{cases}$$

as a final state for the conservation law and

$$w_T(x) = \begin{cases} 0.1, & \text{if } x \leq -1, \\ 0.25(x + 1) + 0.1, & \text{if } -1 < x \leq 0, \\ 0.25(x - 1) + 0.4, & \text{if } 0 < x \leq 1, \\ 0.4, & \text{if } 1 < x, \end{cases}$$

as a final state for the transport equation. We choose $\Delta x = 7.8 \times 10^{-4}$ and $\Delta t = \Delta x/2$.



$\tilde{v}_0$
v_0
0.8
0.6
0.4
0.2
−1.2 −1 −0.8 −0.6 −0.4 −0.2 0 0.2
x
$\tilde{w}_0$
w_0
0.5
0.4
0.3
0.2
0.1
0
−2 −1.5 −1 −0.5 0 0.5
x
$\tilde{v}_T$
v_T
0.8
0.6
0.4
0.2
−0.6 −0.4 −0.2 0 0.2 0.4 0.6
x
$\tilde{w}_T$
w_T
0.5
0.4
0.3
0.2
0.1
0
−2 −1.5 −1 −0.5 0 0.5 1
x

Unreachable profiles and an optimization problem

we fix $T > 0$ and consider a target profile $v_{\mathrm{tar}} = (v_{\mathrm{tar}}, w_{\mathrm{tar}})$ which is *not attainable* in time T for the system (T). We assume that

$$v_{\mathrm{tar}}(x) = \begin{cases} v^-, & \text{for } x < a, \\ \bar{v}(x), & \text{for } x \in [a,b], \\ v^+, & \text{for } x > b, \end{cases} \qquad w_{\mathrm{tar}}(x) = \begin{cases} w^-, & \text{for } x < a, \\ \bar{w}(x), & \text{for } x \in [a,b], \\ w^+, & \text{for } x > b, \end{cases}$$

for some essentially bounded functions $\bar{v}$, $\bar{w}$.

We want to characterize the initial conditions which drive the system as close as possible to v_{tar} with respect to the L^2 norm. These are the minima of

$$J_0(Q_0) = \|\mathfrak{S}_T^+(Q_0) - v_{\mathrm{tar}}\|_{L^2(\mathbb{R};\mathbb{R}^2)},$$

where $\mathfrak{S}_T^+(Q_0)$ belongs to the subset of $\mathfrak{A}_T(\mathbb{R})$ defined as

$$\mathcal{U}^T(v_{\mathrm{tar}}) = \left\{ \begin{aligned} & Q = (q_1, q_2) \in L^\infty(\mathbb{R};\mathbb{R}^2) : \\ & \qquad q_1 \in \mathcal{A}_T(\mathbb{R}, f),\ \|Q\|_{L^\infty(\mathbb{R})} \le C,\ \text{and } Q - v_{\mathrm{tar}} \in L^1(\mathbb{R}) \end{aligned} \right\}.$$

Equivalent formulation

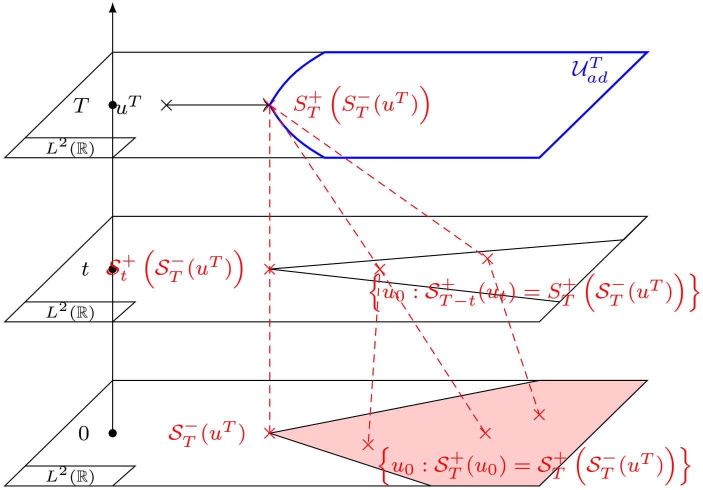
Due to the definition of $\mathfrak{A}_T$, this optimization problem is equivalent to finding $q_{1,\,\mathrm{opt}}$ such that

(Opt) $$\|q_{1,\,\mathrm{opt}} - v_{\mathrm{tar}}\|_{L^2(\mathbb{R})} = \min_{q \in \mathcal{U}_1^T(v_{\mathrm{tar}})} \|q - v_{\mathrm{tar}}\|_{L^2(\mathbb{R})},$$

where the admissible set $\mathcal{U}_1^T(v_{\mathrm{tar}})$ is defined by

$$\mathcal{U}_1^T(v_{\mathrm{tar}}) = \left\{ q \in L^\infty(\mathbb{R}) : \, q \in \mathcal{A}_T\left(\mathbb{R}, v \mapsto \frac{v}{1+v}\right), \; \|q\|_{L^\infty(\mathbb{R})} \le C, \text{ and } \operatorname{supp}(q - v_{\mathrm{tar}}) \subset K \right\}.$$

[Liard–Zuazua, *SIMA* 2023]: $q_{1,\,\mathrm{opt}} = \mathcal{S}_T^+(\mathcal{S}_T^-(v_{\mathrm{tar}}))$.



However, $(q_{1,\,\mathrm{opt}}, w_{\mathrm{tar}})$ does not necessarily belong to $\mathsf{A}_T(\mathbb{R})$ as $q_{1,\,\mathrm{opt}} - w_{\mathrm{tar}}$ is not everywhere positive.

Modified strategy.

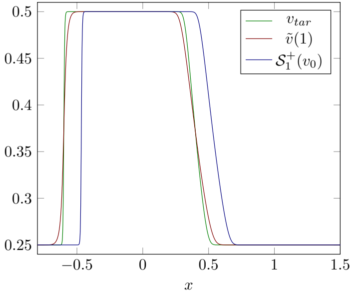
- First, we associate to U_{tar} the profile $v_{\mathrm{tar}} = (v_{\mathrm{tar}} = u_1^{\mathrm{tar}} + u_2^{\mathrm{tar}}, w_{\mathrm{tar}} = u_1^{\mathrm{tar}} - u_2^{\mathrm{tar}})$.
- Then we apply the strategy above to find $q_{1,\,\mathrm{opt}}$.
- Finally, we consider $q_{2,\,\mathrm{opt}} = \min\{q_{1,\,\mathrm{opt}}, w_{\mathrm{tar}}\}$ so to obtain a second component which is as close as possible to w_{tar}, under the constraint that $q_{1,\,\mathrm{opt}} - q_{2,\,\mathrm{opt}} \geq 0$.

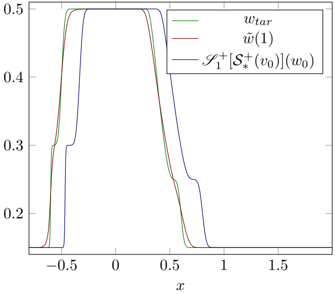
The couple $Q_{\mathrm{opt}} = (q_{1,\,\mathrm{opt}}, q_{2,\,\mathrm{opt}})$ is in $\mathsf{A}_T(\mathbb{R})$, so that the profile attainable at time T for (C) which is the closest to U_{tar} in L^2 is

$$U_{\mathrm{opt}} = \left(\frac{1}{2}(q_{1,\,\mathrm{opt}} + q_{2,\,\mathrm{opt}}),\ \frac{1}{2}(q_{1,\,\mathrm{opt}} - q_{2,\,\mathrm{opt}}) \right).$$

Given the initial conditions $V_0 = (v_0, w_0)$, with

$$v_0(x) = \begin{cases} 0.5, & \text{if } -1 < x < 0, \\ 0.25, & \text{otherwise,} \end{cases} \qquad w_0(x) = \begin{cases} 0.5, & \text{if } -1 < x < 0, \\ 0.15, & \text{otherwise,} \end{cases}$$





Thank you for your attention!

[1] G. M. Coclite, N. De Nitti, C. Donadello, and F. Peru. *Inverse design and boundary controllability for the chromatography system.* HAL-04164795, 2023.