Inverse Problems of Nonlinear Wave Equations

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- Let (M, g) be a Lorentzian $(n+1)$-manifold of signature $(-, +\cdots, +)$. Consider

$$\Box_g u + N(u) = f, \quad u|_{t\leq 0} = 0, \tag{1}$$

where $N(\cdot)$ is nonlinear and $\Box_g := d^* d$ locally reads

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- Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Given a connection 1-form $A \in C^\infty(M; T^*M \otimes \mathfrak{g})$, consider
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where $\Box_{g,A} := (d + A)^*(d + A)$ on $\mathbb{M}^{3+1}$ locally reads
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- Given a time-dependent potential V, consider
$$\Box_{g,A} u + Vu + N(u) = f, \quad u|_{t \leq 0} = 0. \tag{3}$$

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    - the X-ray inversion $\Rightarrow$ the potential V

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  - Melrose-Uhlmann, Guillemin-Uhlmann, Greenleaf-Uhlmann : Such interactions are Fourier Integral Operators associated multiple intersecting Lagrangians.

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- Feizmohammadi-Oksanen : recover V in (3)
- Feizmohammadi-Lassas-Oksanen : introduce the 3-to-1 scattering relation based on the third order linearization to recover g in (1).

Third order linearisation

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- The derivatives
$$\begin{aligned} v_{(j)} &:= \partial_{\epsilon_{(j)}} \phi|_{\epsilon=0} \\ v_{(jk)} &:= \partial_{\epsilon_{(jk)}} \phi|_{\epsilon=0} \\ v_{(123)} &:= \partial_{\epsilon_{(1)}} \partial_{\epsilon_{(2)}} \partial_{\epsilon_{(3)}} \phi|_{\epsilon=0} \end{aligned}$$
satisfy linear wave equations
$$(\Box_g - A \cdot d) v_{(j)} = f_{(j)} \tag{4}$$
$$(\Box_g - A \cdot d) v_{(jk)} = 0 \tag{5}$$
$$(\Box_g - A \cdot d) v_{(123)} = \frac{1}{2} \sum_{\{i,j,k\}=\{1,2,3\}} \langle v_{(i)}, v_{(j)} \rangle v_{(k)}. \tag{6}$$

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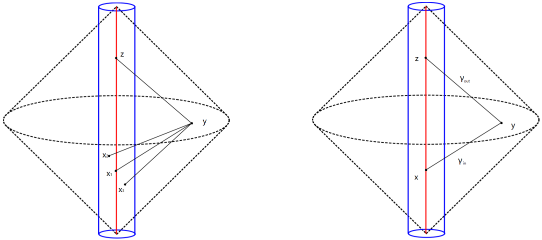

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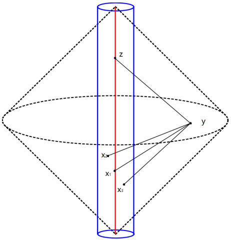

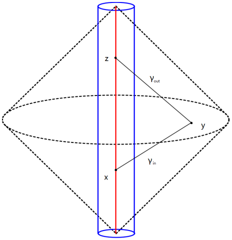

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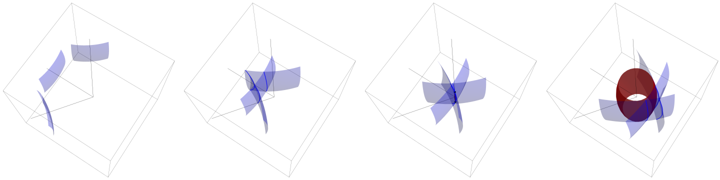

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- From the <u>measurable data</u> $L^{g,A}$, we reconstruct the <u>geometry</u> (g, A) by understanding the <u>X-ray transform</u> $\mathbf{S}^{g,A}$.

$$\begin{array}{ccccc} \text{Measuable data} & \Rightarrow & \text{X-ray transform} & \Rightarrow & \text{Geometry} \\ L^{g_1,A_1} = L^{g_2,A_2} & \Rightarrow & \mathbf{S}^{g_1,A_1} = \mathbf{S}^{g_2,A_2} & \Rightarrow & [g_1, A_1] = [g_2, A_2], \end{array}$$

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$$\|[A] - [B]\|_{C_0(\mathbb{D}\setminus\mho)} \lesssim C_{A,B}\,\tilde{C}_{A,B,\theta,\bar{s}}\,\|A - B\|^{\theta}_{C_0^{\bar{s}/\theta}(\mathbb{D})}\,\|L_A - L_B\|^{1-\theta}_{\mho,s}\,.$$

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Theorem

For any $0 < \theta \leq 1$ *and* $\bar{s} > 3$ *with* $\bar{s}/\theta \in \mathbb{Z}$,

$$\|[A] - [B]\|_{C_0(\mathbb{D}\setminus\mho)} \lesssim C_{A,B}\,\tilde{C}_{A,B,\theta,\bar{s}}\,\|A - B\|^{\theta}_{C_0^{\bar{s}/\theta}(\mathbb{D})}\,\|L_A - L_B\|^{1-\theta}_{\mho,s}\,.$$

- C : Reduction via the third order linearization and the 3-to-1 scattering relation to the stability of the broken X-ray

Theorem

$$\left\|\mathbf{S}^A_{z_{\min}\leftarrow\boldsymbol{y}\leftarrow\boldsymbol{x}} - \mathbf{S}^B_{z_{\min}\leftarrow\boldsymbol{y}\leftarrow\boldsymbol{x}}\right\|_{C_0(\mathcal{F}^X)} \lesssim \|L_A - L_B\|_{\mho,s}\,,$$

$$\|\rho([A]) - \rho([B])\|_{C_0(\mathbb{D}\setminus\mho)} \lesssim \tilde{C}_{A,B}\left\|\mathbf{S}^A_{z\leftarrow\boldsymbol{y}\leftarrow\boldsymbol{x}} - \mathbf{S}^B_{z\leftarrow\boldsymbol{y}\leftarrow\boldsymbol{x}}\right\|_{C_0^1(\mathcal{F}^X)}\,.$$

Thank you for your attention.