On the construction of exact control for the wave equation

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contents

Control of wave equation

Let $T > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. $\omega \neq \emptyset$ is an open subset of Ω, $\Gamma_1 \neq \emptyset$ is an open subset of $\partial\Omega$. χ_ω and χ_{Γ_1} denote the characteristic functions of ω and Γ_1.
Consider the following wave system:

$$(1) \qquad \begin{cases} y_{tt} - \Delta y = \chi_\omega u, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x), & x \in \Omega. \end{cases}$$

Exact controllability problem:
Given initial data (y_0, y_1) and target data (y^0, y^1) in some space, can we choose u in suitable space such that

$$y(T,x) = y^0(x), y_t(T,x) = y^1(x), x \in \Omega?$$

Exact null controllability problem:
Given initial data (y_0, y_1) in some space, can we choose u in suitable space such that

$$y(T,x) = 0, y_t(T,x) = 0, x \in \Omega?$$

Observability

The controllability of (1) is equivalent to the observability of the following system

$$(2)\qquad \begin{cases} v_{tt} - \Delta v = 0, & (t,x) \in (0,T) \times \Omega, \\ v(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ v(T,x) = v^0(x), \ \ v_t(T,x) = v^1(x), & x \in \Omega. \end{cases}$$

The observability inequality:

$$\frac{1}{2}\Big(\|v^1\|^2_{L^2(\Omega)} + \|\nabla v^0\|^2_{L^2(\Omega)}\Big) \leqslant D \int_0^T \|v_t\|^2_{L^2(\omega)} dt.$$

Multiplier geometric condition(MGC)

Assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and Γ_0, Γ_1 are nonempty. Furthermore there exists an x_0 such that

$$(x - x^0) \cdot \nu(x) < 0, \forall x \in \Gamma_0$$

$$(x - x^0) \cdot \nu(x) \geq 0, \forall x \in \Gamma_1$$

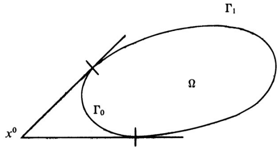

Figure: example

Geometric Control Condition(GCC)

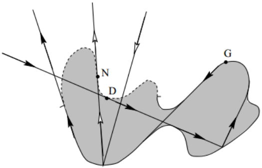

Figure: GCC condition (L. Miller)

There exists $T > 0$ such that every geodesic traveling at speed 1 meets ω or Γ_1 (except for diffractive points) in $(0, T)$.

Theorem (Bardos-Lebeau-Rauch (1992))

Assume that GCC holds. Then the system (1) *is exactly controllable on* $L^2 \times H^{-1}$.

Question: How to construct an exact control?

HUM

Theorem (J.-L. Lions)

Assume that Ω satisfies MGC. The system (1) is exactly controllable on $L^2 \times H^{-1}$.

The control u satisfies

$$(3) \qquad \begin{cases} u_{tt} - \Delta u = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(T,x) = u^0(x), \ \ u_t(T,x) = u^1(x), & x \in \Omega. \end{cases}$$

J.-L. Lions use a solution of wave equation to control (1), taking the final data to minimize the cost.

Linearization

Consider the following control problem of damped linear wave equation

$$(4) \quad \begin{cases} y_{tt} + y_t - \Delta y = \chi_\omega u, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & x \in \Omega. \end{cases}$$

We give a new method to prove the null controllability of system (4).

Contraction mapping principle

We take $u = 2\chi_\omega \cdot z_t$, where z satisfies

$$(5)\quad \begin{cases} z_{tt} - z_t - \Delta z = 0, & (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ z(T,x) = z^0(x),\ z_t(T,x) = z^1(x), & x \in \Omega. \end{cases}$$

Then we define a map

$$\mathcal{F} : (z^0, z^1) \mapsto \big(y(T) + z^0, -y_t(T) + z^1\big).$$

By the observability inequality, we can prove that $\mathcal{F}$ is a contraction map in the set

$$\left\{ \left\|(z^0, z^1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant M(D)\left\|(y_0, y_1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \right\}$$

for some constant $M = M(D)$ large enough. So it has a fixed point, satisfying $y(T) = 0,\ -y_t(T) = 0$,

Since that we can find (z^0, z^1) such that $y(T) = 0,\ -y_t(T) = 0$, then u is the desired control function.

Galerkin method

Firstly, we take the standard orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$, such that

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j \\ \varphi_j|_{\partial\Omega} \equiv 0 \end{cases}$$

Let $y_N = \sum_{j=1}^{N} g_{jN}(t)\varphi_j, \quad v_N = \sum_{j=1}^{N} h_{jN}(t)\varphi_j$ satisfy the following initial value problems

$$(6) \qquad \begin{cases} \left(\partial_t^2 y_N - \Delta y_N + \partial_t y_N - \chi\cdot\partial_t v_N, \varphi_i\right)_{L^2} = 0, \\ t=0 : g_{jN} = (y_0,\varphi_j)_{L^2},\ g'_{jN} = (y_1,\varphi_j)_{L^2} \end{cases}$$

$$(7) \qquad \begin{cases} \left(\partial_t^2 v_N - \Delta v_N - \partial_t v_N, \varphi_i\right)_{L^2} = 0, \\ t=T : h_{jN} = a_j,\ h'_{jN} = b_j \end{cases}$$

So we can define the following map

$$\tilde{\mathcal{F}} : \big(v_N(T), \partial_t v_N(T)\big) \mapsto \big(y_N(T), \partial_t y_N(T)\big) \tag{8}$$

or equivalently

$$\begin{aligned}\mathcal{F} : \big(a_1, \cdots, a_N, b_1, \cdots, b_N\big) \\ \mapsto \big(g_{1N}(T), \cdots, g_{NN}(T), g'_{1N}(T), \cdots, g'_{NN}(T)\big)\end{aligned} \tag{9}$$

By the well-posedness result of linear system, the above map is continuous from $\mathbb{R}^{2N}$ to itself.
Goal:to prove that $\mathcal{F}$ has a zero point.

By Brouwer fixed point theorem, we can prove the following lemma.

Lemma

If there exists $r > 0$, such that the continuous map $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$x \cdot \mathcal{F}(x) \geqslant 0, \qquad \forall\ |x| = r$$

then there exists $x_0 \in B_r$ s.t. $\mathcal{F}(x_0) = 0$.

We need to estimate

$$J_1 \triangleq \int_\Omega \partial_t y_N(T)\partial_t v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx$$

Denote $E(u(t)) = \|u(t)\|^2_{H^1_0} + \|u_t(t)\|^2_{L^2}$, by the standard energy estimate of y_N and v_N, as well as the observability inequality

$$\frac{1}{2}\Big(\|v_t(T)\|^2_{L^2(\Omega)} + \|\nabla v(T)\|^2_{L^2(\Omega)}\Big) \leqslant D\int_0^T \|v_t\|^2_{L^2(\omega)}dt$$

we obtain that

Property

If $E(v_N(T))$ is large enough, then $J_1 \geqslant 0$.

which means $x \cdot \mathcal{F}(x) \geqslant 0$ in the lemma, so $\mathcal{F}$ has a zero point.

By energy estimate, $\{\partial_t v_N\}_{N=1}^{\infty}$ is bounded in $L^2(0,T;H^1(\omega))$, thus has a subsequence that converges weakly, whose limit is the desire control function. On the other hand,

$$\{y_N\}_{N=1}^{\infty} \subset L^{\infty}(0,T;H_0^1(\Omega))$$
$$\{\partial_t y_N\}_{N=1}^{\infty} \subset L^{\infty}(0,T;L^2(\Omega))$$

is bounded too, and converge to the solution. Thus we have the null controllability.

Consider the following semilinear wave system:

$$(10) \quad \begin{cases} y_{tt} - \Delta y = f(y) + \chi_\omega u, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = \chi_{\Gamma_1} h(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x), \ y_t(0,x) = y_1(x), & x \in \Omega. \end{cases}$$

Internal control ($h \equiv 0$):

- If $f(s)$ behaves like $-slog^p(1+|s|), 1 \leq p \leq 2$ as $|s| \to \infty$, the system is exactly controllable (see E. Zuazua(1993) for 1-D; X. Fu, J. Yong and X. Zhang (2007) for n-D)
- If $f(y) = -y^p, n = 3$, the system is exactly controllable when $\omega = \Omega / B(x_0, r)$ (B. Dehman, G. Lebeau, E. Zuazua (2003) for $p < 5$ or C. Laurent (2011) for $p = 5$).

Boundary control ($u \equiv 0$):

- If $f \in W^{1,\infty}_{loc}(\mathbb{R})$ is a locally Lipschitz function, the system is exactly controllable (E. Zuazua (1990)).

Consider the following internal control problem

$$
(11)\qquad \begin{cases} y_{tt} + f(y_t) - \Delta y = \chi_\omega u, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x), \ \ y_t(0,x) = y_1(x) & x \in \Omega. \end{cases}
$$

Theorem (Y. Cui, P. Lu and Z (preprint))

Let $f \in C(\mathbb{R})$ *satisfies* $f(0) = 0,\ \big|f(a) - f(b)\big| \leqslant L|a-b|$, *and there exists* $L > \tilde{L} > 0$, *s.t.,*

$$(a-b)\big(f(a) - f(b)\big) \geqslant \tilde{L}(a-b)^2.$$

Assume that ω *satisfies GCC, then* (11) *is null controllable provided that* $2D(L-\tilde{L})^2 < L\tilde{L}^2$.

The proof is based on Galerkin method.

Quasilinear wave equation, boundary control

- see T. Li and L. Yu (06) for 1-D case;
- see Z. Lei and Z (08) or Yao (2010) for 2, 3-D case;

Main result

Consider the following system

$$
\begin{cases}
y_{tt} - \Delta y = F(y, y_t, \nabla y, \nabla^2 y) + \chi_\omega u(t,x), & (t,x) \in (0,T) \times \Omega \\
y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\
y(0,x) = y_0(x),\ \ y_t(0,x) = y_1(x) & x \in \Omega.
\end{cases} \tag{12}
$$

Theorem (Y. Cui, P. Lu and Z (preprint))

Assume that $F(y, y', y'') = O(|y|^2 + |y'|^2 + |y''|^2)$ *and* Ω *satisfies MGC,* $\omega = \Omega \cap O_\epsilon(\Gamma_1)$, *then the system* (12) *is locally null controllable.*

The proof is based on Picard iteration (Contraction mapping principle).

Consider the following damped wave system

$$
\begin{cases} y_{tt} - \Delta y + y_t = F(y, y', y") + \chi_\omega(x) u(t, x), & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x), \ y_t(0, x) = y_1(x) & x \in \Omega. \end{cases} \tag{13}
$$

Our goal is to obtain the controllability of (13) now!

Case without damping

Consider the internal control problem of linear wave equation

$$
\begin{cases}
y_{tt} - \Delta y = \chi_\omega(x) u(t,x), & (t,x) \in (0,T) \times \Omega, \\
y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\
y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & x \in \Omega.
\end{cases}
$$

Let $y = e^t \tilde{y}$ and $u = e^t \tilde{u}$, then the system is transformed into a system with damping.

Locally controllability of quasi-linear system

In the quasi-linear case, firstly we write

$$\begin{cases} y_{tt} + b_0 y_t - \sum_{i,j=1}^{n} a_{ij} y_{x_i x_j} = \tilde{b} y + \sum_{i=1}^{n} b_i y_{x_i} + \chi u, \\ y(0,x) = y_0(x), \ y_t(0,x) = y_1(x), \\ y(t,x) = 0, \qquad x \in \ \partial\Omega, \end{cases}$$

where χ is a smooth cut-off function on ω, $\tilde{b}, b_0, a_{ij}, b_i$ are functions of $y, y_t, \nabla y$, satisfying

$$a^{ij} - \delta_{ij}, \ b_0 - 1, \ b_k, \ \tilde{b} \in X_{C,s,\varepsilon},$$

where

$$X_{C,s,\varepsilon} = \Big\{ f \in L^\infty(0,T; L^2(\Omega)) : \big\| \partial_t^j \nabla^k f(t) \big\|_{L^2(\Omega)} \leqslant C\varepsilon, \\ \forall \ j,k \in \mathbb{N}, \ j+k \leqslant s, \ \forall \ t \in [0,T] \Big\},$$

Picard iteration

We establish the following iteration scheme: Put $(z^{(0)}, y^{(0)}) \equiv 0$, knowing $(z^{(\alpha-1)}, y^{(\alpha-1)})$, we define $(z^{(\alpha)}, y^{(\alpha)})$ as follows

$$\begin{cases} y_{tt}^{(\alpha)} + b_0^{(\alpha-1)} y_t^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha-1)} y_{x_i x_j}^{(\alpha)} - \tilde{b}^{(\alpha-1)} y^{(\alpha)} - \sum_{i=1}^{n} b_i^{(\alpha-1)} y_{x_i}^{(\alpha)} \\ = -2\chi \cdot z_t^{(\alpha)}(t), \\ y^{(\alpha)}(0,x) = y_0(x),\ y_t^{(\alpha)}(0,x) = y_1(x), \\ y^{(\alpha)}(t,x) \equiv 0, \qquad x \in \partial\Omega. \end{cases}$$

$$\begin{cases} z_{tt}^{(\alpha)} - b_0^{(\alpha-1)} z_t^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha-1)} z_{x_i x_j}^{(\alpha)} - \tilde{b}^{(\alpha-1)} z^{(\alpha)} - \sum_{i=1}^{n} b_i^{(\alpha-1)} z_{x_i}^{(\alpha)} = 0, \\ z^{(\alpha)}(T,x) = y^{(\alpha-1)}(T,x) + z^{(\alpha-1)}(T,x), \\ z_t^{(\alpha)}(T,x) = y_t^{(\alpha-1)}(T,x) + z_t^{(\alpha-1)}(T,x), \\ z^{(\alpha)}(t,x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\tilde{b}^{(\alpha-1)}, b_0^{(\alpha-1)}, a_{ij}^{(\alpha-1)}, b_i^{(\alpha-1)}$ are functions of $y^{(\alpha-1)}, y_t^{(\alpha-1)}$, $\nabla y^{(\alpha-1)}$.

The order is

$$(z^{(0)}, y^{(0)}) \to \cdots \to z^{(\alpha-1)} \to y^{(\alpha-1)} \to z^{(\alpha)} \to y^{(\alpha)} \to \cdots$$

Consider the internal control problem of linear heat equation

$$\begin{cases} y_t - \Delta y = \chi_\omega(x) u(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x) \neq 0, & x \in \Omega. \end{cases}$$

Our goal: to find a function u such that $y(T) \equiv 0$.

We choose $u = \frac{D^{\frac{1}{2}} \|y_0\|_{L^2} v}{(\int_0^T \int_\omega |v|^2 dx dt)^{\frac{1}{2}}}$, where

$$\begin{cases} v_t + \Delta v = 0, & (t,x) \in (0,T) \times \Omega, \\ v(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ v(T,x) = v^0(x) \neq 0, & x \in \Omega, \end{cases}$$

satisfies the observability inequality

$$\|v(0)\|_{L^2}^2 \leqslant D \int_0^T \int_\omega |v|^2 dx dt.$$

Thanks for your attention!