Optimal Sampling for Linear Control Systems

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In digital control systems, the state is sampled at given sampling instants and the control is kept constant between two consecutive instant. The choice of sampling instants is crucial in order to perform an optimal control and to govern efficiently the system.

Reducing the number of sampling instants in digital controllers has a great beneficial impact on the system features: the computing power required by the controller, the amount of needed communication bandwidth, the energy consumed by the controller, etc.

By optimal sampling problem we mean the selection of sampling instants and control inputs, such that a given function of the state and control is minimized.

We consider the optimal sampling problem in the framework of linear quadratic regulator problem. More precisely, we consider a linear control system

$$x' = Ax + Bu \text{ in } [0, T], \qquad x(0) = x_0$$

and the goal is to optimize a quadratic quantity of the form

$$\int_0^T (Qx \cdot x + Ru \cdot u)\, dt + Sx(T) \cdot x(T).$$

Here $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and A, B, Q, R, S are matrices of the corresponding dimensions. As usual in linear quadratic control problems, we assume Q, S positive semidefinite, and R positive definite.

The link between optimal state and optimal control is provided by the Riccati differential equation

$$K' = KBR^{-1}B^tK - A^tK - KA - Q, \qquad K(T) = S$$

which gives the relation

$$u(t) = -R^{-1}B^tK(t)x(t)$$

In this way, the optimal value of the cost is

$$J_\infty = K(0)x_0 \cdot x_0.$$

A sequence $0 = t_0 < t_1 < \cdots < t_N = T$ is called sampling pattern, and the t_k are called sampling instants. In the interval (t_k, t_{k+1}) between two sampling instants, the control variable is kept constant, equal to a suitable value u_k. For a given sampling pattern, the optimal values u_k that minimize the cost functional can be analytically determined through a discretization process.

Our aim is to compare the efficiency of various sampling methods, finding if possible the optimal one. To do this, for a given sampling method, we introduce the normalized cost

$$c_N = \frac{N^2}{T^2} \frac{J_N - J_\infty}{J_\infty}$$

where N is the number of samples and J_N, J_∞ the minimal discrete cost and the minimal continuous cost respectively. The asymptotic normalized cost will be then

$$c = \lim_{N \to \infty} c_N.$$

The asymptotic normalized cost c of a given sampling method is very convenient to estimate the number of samples to achieve a certain cost increase with respect to the continuous-time case. If for a given sampling method we can tolerate, at most, a (small) factor ε of cost increase with regard to the continuous-time optimal controller, then we need at least

$$N = T(c/\varepsilon)^{1/2}$$

samples, with a cost

$$J_N = J_\infty + N^{-2} J_\infty T^2 c + o(N^{-2}).$$

It is also convenient to introduce the sampling density

$$\sigma_N(t) = \frac{1}{N(t_{k+1} - t_k)} \qquad \forall t \in [t_k, t_{k+1}],$$

and the asymptotic sampling density

$$\sigma(t) = \lim_{N \to \infty} \sigma_N(t).$$

We describe now some sampling methods in terms of the efficiency coefficient c and of the asymptotic density σ.

The Riemann sampling

The simplest (and almost universally used) sampling method is the one obtained by a periodic sampling, where for every k

$$t_{k+1} - t_k = \frac{T}{N}, \quad \text{hence } t_k = \frac{kT}{N}.$$

The asymptotic density is the constant

$$\sigma_{per}(t) = 1/T.$$

In the scalar case $n = 1$ an explicit computation gives (for T large)

$$c_{per} = \frac{1}{24}\Big(A\sqrt{A^2 + \frac{QB^2}{R}} + A^2 + \frac{QB^2}{R}\Big).$$

The Lebesgue sampling

This consists in fixing a threshold δ on the optimal continuous-time input u and, after any sampling instant t_k, the next one t_{k+1} is determined such that

$$|u(t_{k+1}) - u(t_k)| = \delta.$$

It is easy to see that in this case the asymptotic density is

$$\sigma_{leb}(t) = \frac{|u'(t)|}{\int_0^T |u'(s)|\, ds}$$

The asymptotic normalized cost c for the Lebesgue sampling is hard to compute explicitly.

Taking a different sampling density of the form

$$\sigma_\alpha(t) = \frac{|u'(t)|^\alpha}{\int_0^T |u'(s)|^\alpha \, ds}$$

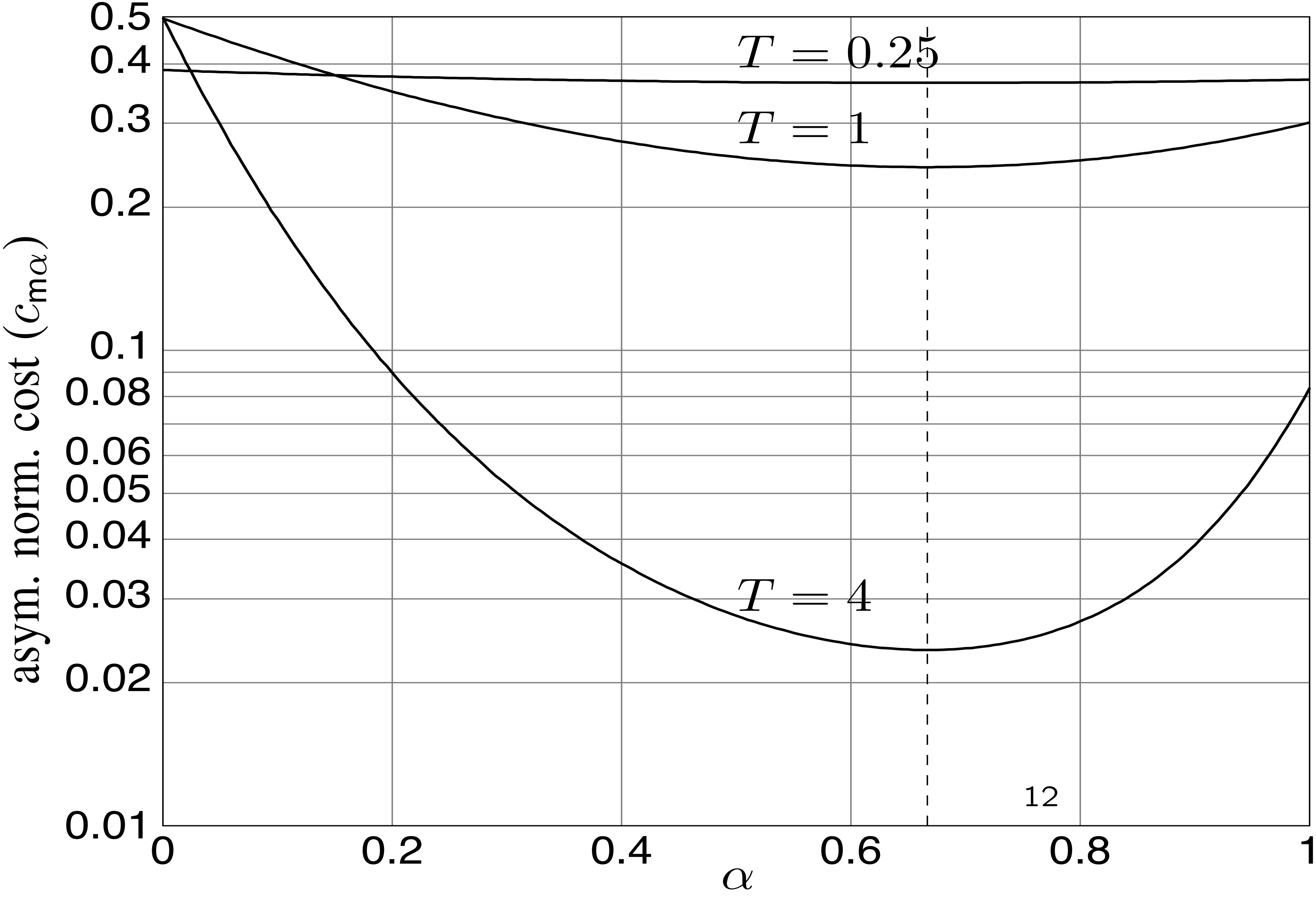
will produce different asymptotic normalized costs c_α: note that the Riemann one corresponds to $\alpha = 0$ and the Lebesgue one to $\alpha = 1$.

The following picture shows the numerical calculation of the asymptotic normalized costs c_α for the linear quadratic problem with

$$A = 1, \quad B = 1, \quad R = 1, \quad Q = 8, \quad S = 4.$$

For the final time T we took three different cases:

$$T = 0.25, \qquad T = 1, \qquad T = 4.$$



0.5
0.4
0.3
0.2
0.1
0.08
0.06
0.05
0.04
0.03
0.02
0.01
asym. norm. cost ($c_{\mathrm{m}\alpha}$)
0
0.2
0.4
0.6
0.8
1
α
$T = 0.25$
$T = 1$
$T = 4$

In order to better understand the picture above, let us consider a different problem: given a function $u(t)$ find the best $L^2(0,T)$ approximation of u by means of piecewise constant functions with a given number N of pieces. We then want to minimize

$$\int_0^T |u(t) - u_N(t)|^2 \, dt$$

where u_N is piecewise constant with N pieces.

We may reformulate this problem as a **mass transportation problem**:

If $\mu = u^{\#}\mathcal{L}$ is the image measure of the Lebesgue measure $\mathcal{L}$ on $[0, T]$ we want to find the sum μ_N of N Dirac masses (with total mass T) which best approximates μ in the sense of Wasserstein:

$$\min \Big\{ W_2^2(\mu, \mu_N) \ : \ \#(\mathrm{spt}\, \mu_N) = N \Big\}.$$

This problem is well known as location problem and as $N \to \infty$ we have:

- The optimal constants u_k are the averages

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_{k+1}}^{t_k} u \, dt.$$

- The asymptotic cost is, if μ is in L^1,

$$W_2^2(\mu, \mu_N) \approx \frac{1}{12N^2} \Big(\int \mu^{1/3} dy \Big)^3.$$

- The asymptotic density of values u_k is

$$\rho(y) = \mu^{1/3}(y) \Big/ \Big(\int \mu^{1/3} dy \Big).$$

- The asymptotic sampling density is

$$\sigma(t) = |u'|^{2/3}(t) \Big/ \Big(\int |u'|^{2/3} dt \Big).$$

Therefore, for the best piecewise constant approximation of a given function $u(t)$ the optimal sampling density has to be proportional to $|u'|^{2/3}$.

We apply now this empirical result to provide an efficient sampling method for optimal control problems.

The empirical method

1. Solve the continuous problem and get the optimal continuous control $u(t)$.

2. Apply to the function $u(t)$ the method to find the best piecewise constant approximation; this gives the sampling density proportional to $|u'|^{2/3}$.

3. Use the same sampling density for the optimal control problem.

QUESTION: How good is this method?

Here are some numerical computations where $T = 1$ and $A = B = R = 1$, $Q = 8$, $S = 4$. We compare the normalized costs for the methods:

- periodic, indicated by $c_{N,per}$
- Lebesgue, indicated by $c_{N,leb}$
- heuristic with $|u'|^{2/3}$, indicated by $c_{N,23}$
- optimal, indicated by $c_{N,opt}$

The number of samples has been taken:

4, 8, 16, 32, 64, 128, 256, 512.

N	$c_{N,per}$	$c_{N,leb}$	$c_{N,23}$	$c_{N,opt}$
4	0.4958	0.3200	0.2541	0.2536
8	0.4980	0.3082	0.2454	0.2454
16	0.4986	0.3033	0.2432	0.2432
32	0.4987	0.3017	0.2426	0.2426
64	0.4987	0.3012	0.2425	0.2425
128	0.4988	0.3011	0.2424	0.2424
256	0.4988	0.3010	0.2424	0.2424
512	0.4988	0.3010	0.2424	0.2424
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
∞	$\frac{1-e^{-6}}{2}$	$\frac{(1-e^{-3})^2}{3}$	$\frac{3(1-e^{-2})^3}{8}$	$\frac{3(1-e^{-2})^3}{8}$
$\approx$	0.4988	0.3010	0.2424	0.2424
$\varepsilon = 2\%$	$N \geq 4.99$	$N \geq 3.88$	$N \geq 3.48$	$N \geq 3.48$

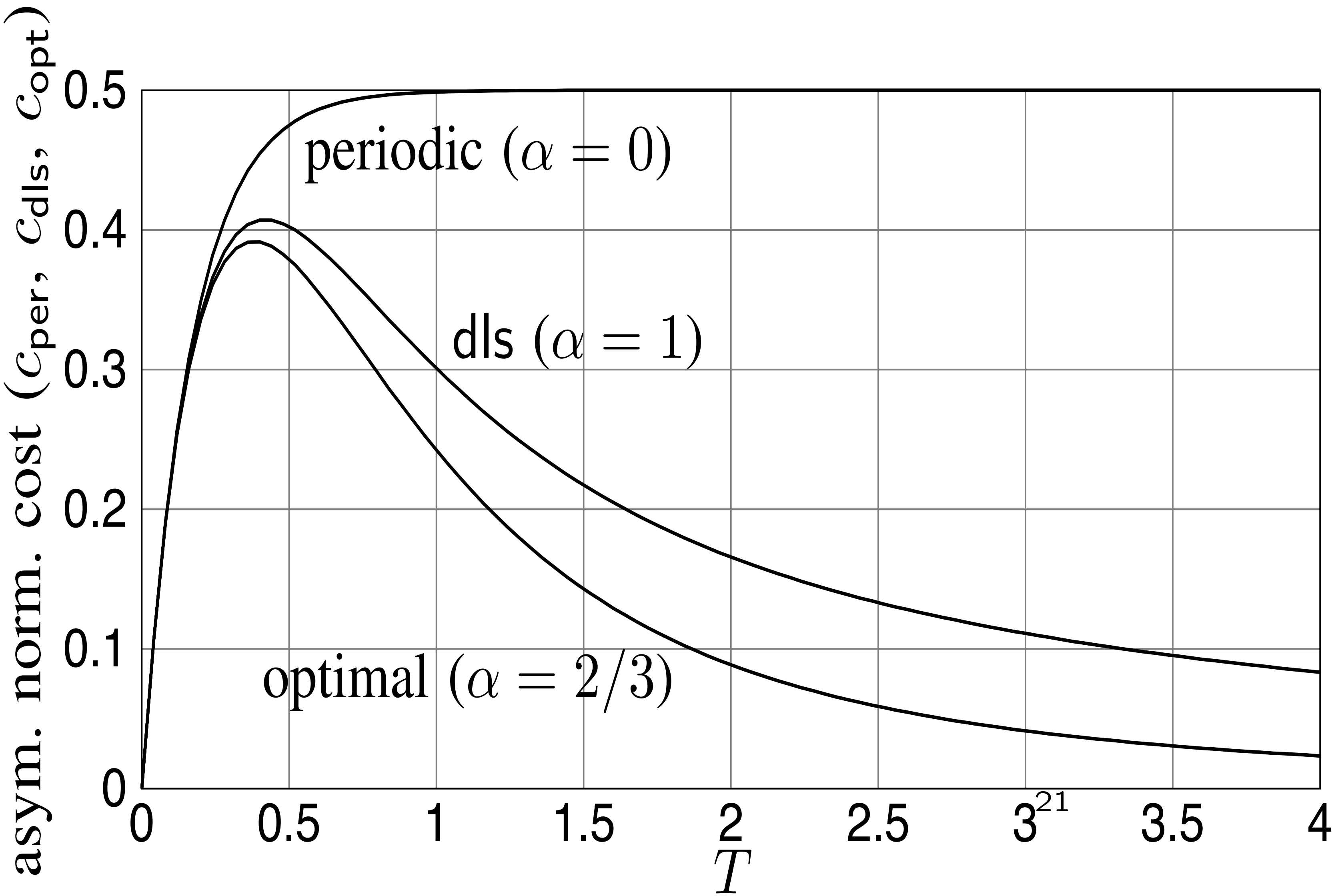
Other examples show a similar behavior.

The asymptotic normalized costs can be explicitly computed for every T and we find:

$$c_{per} = \frac{1 - e^{-6T}}{2}$$

$$c_{leb} = \frac{(1 - e^{-3T})^2}{3T}$$

$$c_{23} = c_{opt} = \frac{3(1 - e^{-2T})^3}{8T^2}$$


periodic ($\alpha = 0$)
dls ($\alpha = 1$)
optimal ($\alpha = 2/3$)
asym. norm. cost (c_{per}, c_{dls}, c_{opt})
T
0
0.1
0.2
0.3
0.4
0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

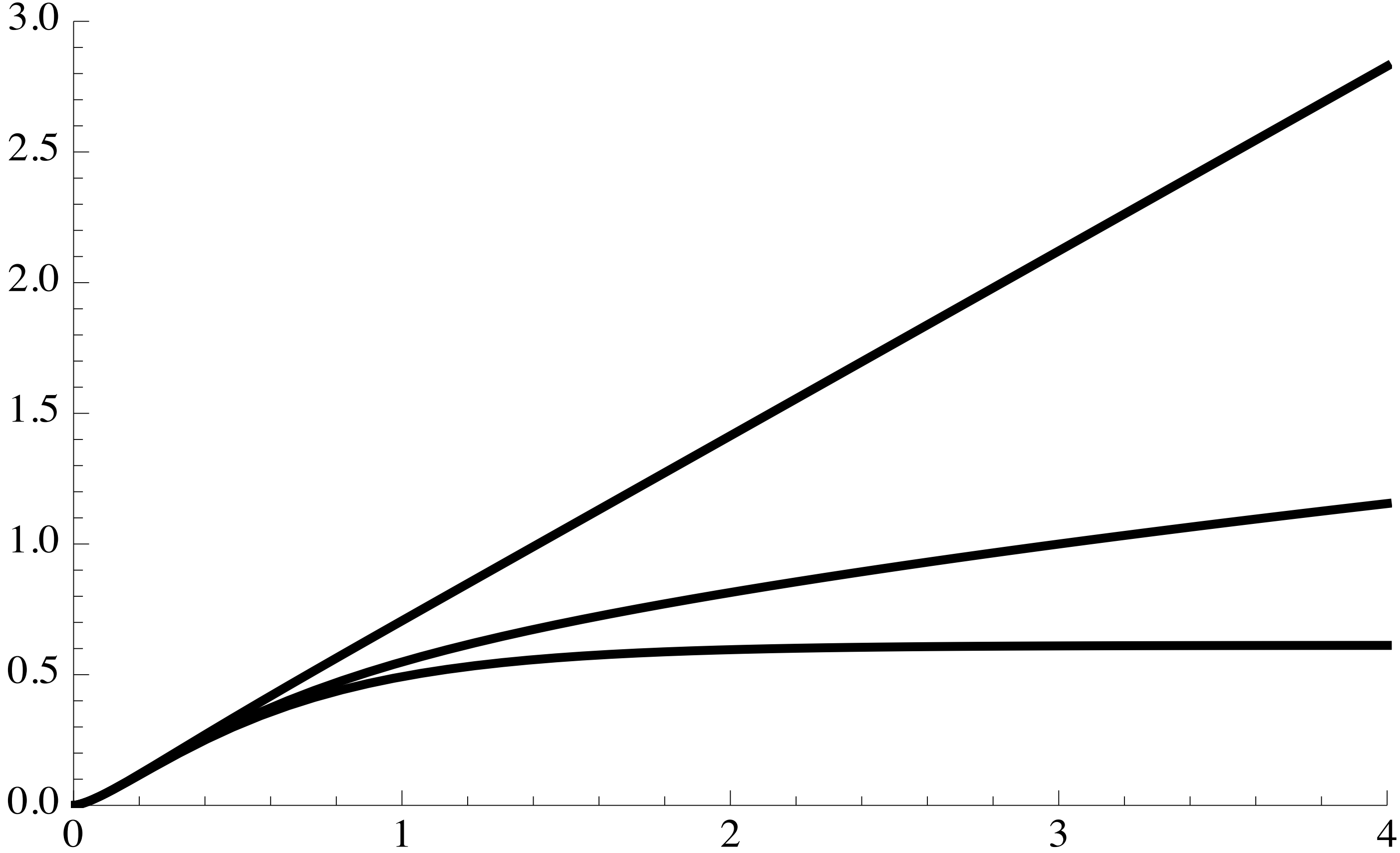
In the same way we can estimate, in terms of T, the minimal number of samples needed to reach a tolerance ε with respect to the continuous cost:

$$N_{per} = \varepsilon^{-1/2} T \sqrt{\frac{1-e^{-6T}}{2}} \approx \varepsilon^{-1/2} \frac{T\sqrt{2}}{2}$$

$$N_{leb} = \varepsilon^{-1/2} T \sqrt{\frac{(1-e^{-3T})^2}{3T}} \approx \varepsilon^{-1/2} \frac{\sqrt{3T}}{3}$$

$$N_{23} = N_{opt} = \varepsilon^{-1/2} T \sqrt{\frac{3(1-e^{-2T})^3}{8T^2}} \approx \varepsilon^{-1/2} \frac{\sqrt{6}}{4}$$

Note that N_{23} does not depend on T for T large, whereas N_{per} grows as T and N_{leb} as $\sqrt{T}$.



Plot of $\varepsilon^{1/2}N_{per}$, $\varepsilon^{1/2}N_{leb}$, $\varepsilon^{1/2}N_{opt}$ in terms of T