Data Assimilation for Gas Flows on Networks

Jan Giesselmann, Teresa Kunkel, Varun Kumar (Darmstadt), Martin Gugat (Erlangen)

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TRR 154

SPP 2410 CoScaRa

State estimation on networks using observers

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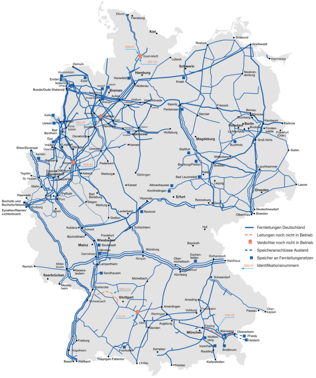

- Goal: Estimate the current system state in gas/H2 networks (pressure, velocity in all pipes) to improve control decisions.
- The state can only be measured at a certain number of points.
- Combine model/simulation and measurements by constructing an observer system, i.e. IBVP that uses approximate initial data and nodal measurements.
- How many measurement points are needed so that we can guarantee synchronization, i.e the observer state converges to the true system state, for long times?
- How can we handle discrepancies between our model and the true evolution?

Overview

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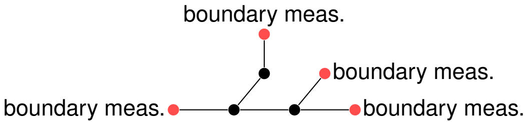
Minimal number of measurement points

Model discrepancies

Setup

We consider the case of

- full state measurements on all boundary nodes (no inner nodes); no other measurement points;
- no measurement errors;
- original system and observer system coincide;
- linear model: wave equation (without friction), i.e. linearized Euler equations. Coupling conditions correspond to conservation of mass and continuity of enthalpy or pressure in the Euler equations.
- We use the measured state as boundary condition for the observer system.



PDEs and coupling conditions

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On each pipe (edge on the graph) the model reads

$$\begin{pmatrix} R_+ \\ R_- \end{pmatrix}_t + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} R_+ \\ R_- \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $c > 0$ is the wave speed.
We consider delta-prime coupling conditions at inner nodes

$$\sum_{e \in \mathcal{E}(v)} (R^e_{\text{out}}(t,v) - R^e_{\text{in}}(t,v)) = 0, \quad R^e_{\text{out}}(t,v) + R^e_{\text{in}}(t,v) = R^f_{\text{out}}(t,v) + R^f_{\text{in}}(t,v) \ \forall e, f \in \mathcal{E}(v)$$

that can be rewritten as

$$R^e_{\text{out}}(t,v) = -R^e_{\text{in}}(t,v) + \frac{2}{|\mathcal{E}(v)|} \sum_{g \in \mathcal{E}(v)} R^g_{\text{in}}(t,v), \quad t \in (0,T), v \in \mathcal{V} \setminus \mathcal{V}_\partial$$

where $\mathcal{E}(v)$ is the set of edges adjacent to some node v.
We prescribe R_{out} on each boundary node.
$R_{\text{in}}, \ R_{\text{out}}$ is from the viewpoint of the nodes and corresponds to $R_\pm$ depending on geometry.

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Evolution equation of difference system

We denote the difference $\delta_\pm$ between the solutions of the original system and the observer system. It satisfies

$$
\begin{aligned}
&\partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} = 0, && e \in \mathcal{E}, \\
&\delta_\pm^e(0, x) = y_\pm^e(x) - z_\pm^e(x), && x \in (0, \ell^e), e \in \mathcal{E}, \\
&\delta_{\text{out}}^e(t, v) = 0, && t \in (0, T), v \in \mathcal{V}_\partial, e \in \mathcal{E}(v), \\
&\delta_{\text{out}}^e(t, v) = -\delta_{\text{in}}^e(t, v) + \tfrac{2}{|\mathcal{E}(v)|} \sum_{g \in \mathcal{E}(v)} \delta_{\text{in}}^g(t, v), && t \in (0, T), v \in \mathcal{V} \setminus \mathcal{V}_\partial.
\end{aligned}
$$

where V_∂ is the set of boundary nodes and $y_\pm^e$ and $z_\pm^e$ are the initial data of the original system and the observer system, respectively.

Inner cycles prevent synchronization

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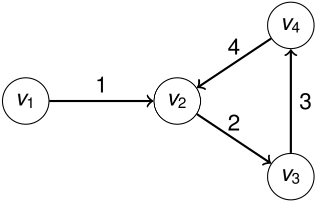
Lemma

If the graph $G = (\mathcal{V}, \mathcal{E})$ contains a cycle consisting of inner points then synchronization cannot be guaranteed, i.e. there exist initial data $y_\pm, z_\pm$ such that

$$\lim_{t\to\infty} \|(\delta_+(t), \delta_-(t))\|_{L^2(\mathcal{E})} \neq 0.$$

Proof.

We consider an example. The general case can be handled analogously.



For any $a \in \mathbb{R}$ a stationary solution is given by
$\delta^1_\pm = 0, \delta^j_+ = a, \delta^j_- = -a$ for $j \in \{2, 3, 4\}$.

$\square$

Observability inequality

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Lemma

Let $G = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network with N inner nodes. Let ℓ_m denote the maximal length of a pipe in G. Then there exists a constant $C > 0$ such that for $T \geq N\frac{\ell_m}{c}$ and $t > T$ we have

$$\|(\delta_+, \delta_-)(t, \cdot)\|^2_{L^2(\mathcal{E})} \leq C \sum_{v \in \mathcal{V}_\partial} \|(\delta_+, \delta_-)(\cdot, v)\|^2_{L^2([t-T, t+T])},$$

Sketch of proof

We use induction in N
For $N = 1$ the graph is star shaped and the result is known.

Observability inequality: induction step

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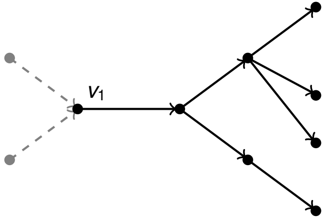
For $N > 1$ there exists an inner node v_1 that has only one edge connected to another inner node.
Let G_1 be the sub-graph obtained by removing from G all edges connecting v_1 to boundary nodes $\Longrightarrow$ observation inequality holds on G_1.



G: gray and black; G_1: in black

Note $\mathcal{V}_\partial(G_1) \subset \{v_1\} \cup \mathcal{V}_\partial(G)$. Thus, we need to bound

$$\|(\delta_+, \delta_-)(\cdot, v_1)\|_{L^2(t-T,t+T)} \leq C \sum_{v \in \mathcal{V}_\partial(G)} \|(\delta_+, \delta_-)(\cdot, v)\|_{L^2(t-T-c,t+T+c)}$$

Synchronization

Theorem

Let $G = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network. Then there exist constants $\mu > 0$, $C_1 > 0$ such that

$$\|(\delta_+, \delta_-)(t, \cdot)\|^2_{L^2(\mathcal{E})} \leq C_1 \exp(-\mu t) \quad \forall t > 0.$$

Sketch of Proof:

$$\|(\delta_+, \delta_-)(t + \tilde{t}, \cdot)\|^2_{L^2(\mathcal{E})} - \|(\delta_+, \delta_-)(t - \tilde{t}, \cdot)\|^2_{L^2(\mathcal{E})}$$

$$\leq (-c) \int_{t-\tilde{t}}^{t+\tilde{t}} \sum_{v \in \mathcal{V}_\partial} \sum_{e \in \mathcal{E}(v)} \left(|\delta^e_{\text{out}}(s, v)|^2 + |\delta^e_{\text{in}}(s, v)|^2 \right) ds$$

$$= (-c) \sum_{v \in \mathcal{V}_\partial} \|(\delta_+, \delta_-)(\cdot, v)\|^2_{L^2([t-\tilde{t}, t+\tilde{t}])}.$$

Apply observability and modified Gronwall lemma.

What to do about general networks?

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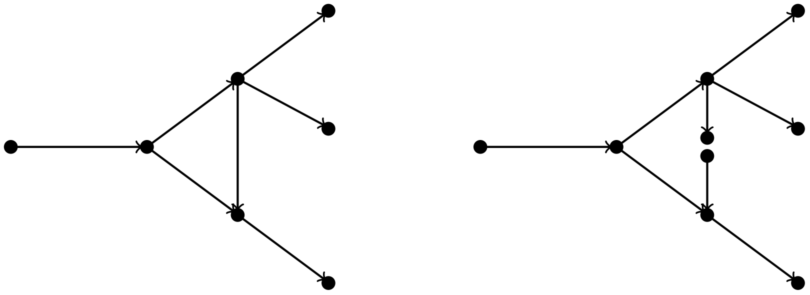
Inserting a full state measurement at a point inside a pipe (edge) into the observer is equivalent to splitting that edge and adding a boundary node for each half.
If we add one measurement per cycle, we end up with a tree shaped graph $\rightarrow$ observer state converges exponentially to the true systems state



What about finite time synchronization?

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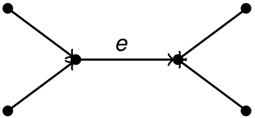
For star shaped networks without friction there is finite time synchronization.

If there is one inner pipe whose end-nodes have more than two adjacent pipes this is no longer true, due to reflection at nodes with more than two adjacent pipes:



If we start with $\delta^e_\pm(0,x) = 1 \;\forall x \in e$ and $\delta_\pm(0,x) = 0$ for all $x \notin e$ then for any $n \in \mathbb{N}$

$$\delta_\pm(n\frac{|e|}{c}, x) = \frac{1}{3^n} \forall x \in e, \quad \delta_-(n\frac{|e|}{c}, x) = \frac{2}{3^n} \;\forall x \notin e$$

$\rightarrow$ (only) exponential decay.

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Setup

True dynamics

$$\begin{pmatrix} R_+ \\ R_- \end{pmatrix}_t + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} R_+ \\ R_- \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observer model

$$\begin{pmatrix} S_+ \\ S_- \end{pmatrix}_t + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} S_+ \\ S_- \end{pmatrix}_x = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$$

with $\epsilon_\pm$ given functions that account for the difference between the models.

We can again study the evolution of the differences $\delta_\pm = R_\pm - S_\pm$. If we have state measurements at all nodes we get

Theorem

There exist $T, C, c > 0$ such that for all $t > T$

$$\delta_+(t)^2 + \delta_-(t)^2 \leq C \left(1 + \int_0^t (\varepsilon_+(s)^2 + \varepsilon_-(s)^2) e^{cs} ds \right) e^{-ct}$$

Summary

- Exponential synchronization of boundary observers for linear wave equations with full state measurements for networks without cycles.
- Exponential synchronization is optimal since finite time synchronization does not hold in general.
- In general, there is no synchronization for networks with cycles $\longrightarrow$ one needs to add one measurement per cycle to ensure synchronization.
- Analogous results hold in the case with linear friction. Technical challenge: Riemann invariants are no longer constant along characteristics and interact constantly.
- Conjecture: analogous results hold for non-linear friction – as long as the friction law is Lipschitz, and for non-linear wave equations – as long as solutions are subsonic.
- We plan to analyse minimal numbers of measurements with nonlinear models and model discrepancies.

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Thank you for your attention!