Singularities in Dynamic Programming: A Paradise of Choices, a Nightmare for Numerics

Piermarco Cannarsa

University of Rome “Tor Vergata”
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ORGANISERS

Harbir Antil and Enrique Zuazua

Outline

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1. Dynamic Programming
2. Propagation of singularities
3. Application to computer graphics

Value function

- (f, A) control process, $T > 0$, $(t, x) \in [0, T] \times \mathbb{R}^n$

$$y(\cdot; t, x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(s) = f\big(y(s), \alpha(s)\big) & s \in [t, T] \\ y(t) = x \end{cases}$$

- A compact
- $\varphi : \mathbb{R}^n \to \mathbb{R}$ lower semicontinuous

Mayer Problem $\boxed{\text{minimize} \quad \varphi\big(y(T; t, x, \alpha)\big) \quad \text{over} \quad \alpha : [t, T] \to A}$

Definition

value function

$$v(t, x) = \inf_{\alpha:[t,T]\to A} \varphi\big(y(T; t, x, \alpha)\big) \qquad \big((t, x) \in [0, T] \times \mathbb{R}^n\big)$$

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Hamilton-Jacobi-Bellman equation

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$$H(x,p) = \max_{a\in A} \; -p\cdot f\big(x,a\big) \qquad \big((x,p)\in R^n\times\mathbb{R}^n\big)$$

Theorem

- A compact
- $\varphi:\mathbb{R}^n\to\mathbb{R}$ *locally Lipschitz*

then v *satisfies*

$$\begin{cases} -\partial_t v(t,x) + H\big(x,\partial_x v(t,x)\big) = 0 & (t,x)\in(0,T)\times\mathbb{R}^n \text{ a.e.} \\ v(T,x) = \varphi(x) & x\in\mathbb{R}^n \end{cases} \qquad (HJB)$$

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want to $\boxed{\text{minimize} \quad \varphi\big(y(T; t, x, \alpha)\big) \quad \text{over} \quad \alpha : [t, T] \to A}$

1. find a solution $u \in C\big([0, T] \times \mathbb{R}^n\big) \cap C^1\big((0, T) \times \mathbb{R}^n\big)$ of

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2. construct a 'nice' map (*feedback*) $a : (0, T) \times \mathbb{R}^n \to A$ such that

$$-\partial_x u(t, x) \cdot f\big(x, a(t, x)\big) = H\big(x, \partial_x u(t, x)\big)$$

3. solve the *closed loop* system

$$\begin{cases} \dot{y}(s) = f\big(y(s), a(s, y(s))\big) & s \in [t, T] \\ y(t) = x \end{cases}$$

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Object of this talk

$\Omega \subset \mathbb{R}^n$ bounded $\qquad \boxed{H(x,u,Du) = 0 \qquad \text{a.e. in } \Omega}$

- $u : \Omega \to \mathbb{R}$ Lipschitz viscosity solution
- $p \mapsto H(x,u,p)$ is convex

Singular set

$$\mathrm{Sing}(u) = \{x \in \Omega \mid \nexists Du(x)\}$$

Examples

1. mechanical systems $\qquad \frac{1}{2}\langle A(x)Du, Du\rangle + V(x) = 0 \qquad (x \in \mathbb{T}^n)$
2. HJB equations in $\mathbb{R}^n$ $\qquad \begin{cases} u_t + H(t,x,\partial_x u) = 0 & (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$
3. HJB equations in spaces of measures

$$\int_{\mathbb{T}^n} H(x, \partial_\mu U[\mu](x))\, d\mu(x) = 0 \qquad (\mu \in \mathcal{P}(\mathbb{T}^n))$$

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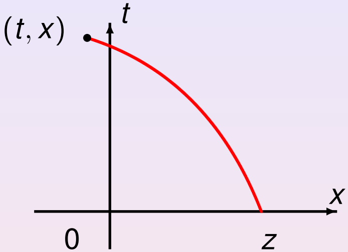
Singularities: a nightmare of numerics

$$\begin{cases} u_t + H(t,x,D_x u) = 0 & (0,T)\times\mathbb{R}^n \\ u(0,x) = u_0(x) & x\in\mathbb{R}^n \end{cases}$$



by using characteristics:
on $[0,T]\times\mathbb{R}^n \setminus \overline{\mathrm{Sing}(u)}$
u is as smooth as the data

$$\begin{cases} \dot{x}(t) = D_p H\big(t, x(t), p(t)\big), & x(0) = z \\ \dot{p}(t) = -D_x H\big(t, x(t), p(t)\big), & p(0) = Du_0(z) \end{cases}$$

Singularities: a paradise of choices

Denote by $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Legendre transform of H

$$L(t, x, q) = \max_{p \in \mathbb{R}^n} \big[\langle q, p\rangle - H(t, x, p)\big]$$

The value function

$$u(t, x) = \inf_{\xi(t)=x} \Big\{ \int_0^t L\big(s, \xi(s), \xi'(s)\big)\, dt + u_0\big(\xi(0)\big)\Big\}$$

gives the viscosity solution of

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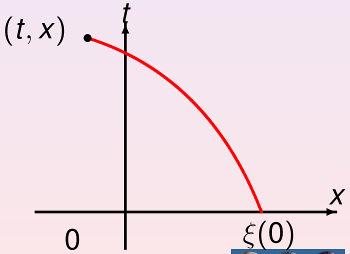

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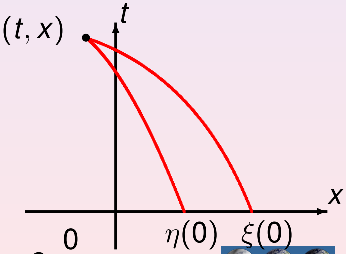

$\exists Du(t,x) \quad \Longleftrightarrow \quad$ unique minimizer

$(t,x) \in \mathrm{Sing}(u) \quad \Longleftrightarrow \quad \#\{\text{minimizers}\} \geqslant 2$

Outline

Generalized characteristics

Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^1$

- $\mathbf{x}_u \in \mathcal{C}^1([0,T];\Omega)$ is a classical characteristic for (u,H), with $u \in \mathcal{C}^1(\Omega)$, if

$$\dot{\mathbf{x}}_u(t) = D_pH\Big(\mathbf{x}_u(t), u\big(\mathbf{x}_u(t)\big), Du\big(\mathbf{x}_u(t)\big)\Big) \quad \text{for all } t \in [0,T)$$

- $\mathbf{x}_u \in \mathrm{Lip}([0,T];\Omega)$ is a generalized characteristic for (u,H), with u semiconcave in Ω, if

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The existence of generalized characteristics follows from general results on differential inclusions [Aubin–Cellina, Springer (1984)]

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Local propagation of singularities

Let

- $u : \Omega \to \mathbb{R}$ be a semiconcave solution $H(x, u, Du) = 0$
- $x_0 \in \mathrm{Sing}(u) \setminus \mathrm{Crit}(u)$ where

$$\mathrm{Crit}(u) = \{x \in \Omega \ : \ 0 \in D_p H(x, u(x), D^+ u(x))\}$$

Theorem (Albano – C 2000, Yu 2006, C – Yu 2009)

There exists a generalized characteristic $\mathbf{x}_u$ on some interval $[0, T)$ such that

$$\mathbf{x}_u(0) = x_0 \quad \& \quad \dot{\mathbf{x}}_u^+(0) \neq 0 \quad \& \quad \mathbf{x}_u(t) \in \mathrm{Sing}(u) \quad \forall t \in [0, T)$$

Global propagation

When can we ensure that $\mathbf{x}_u(t) \in \mathrm{Sing}(u) \quad \forall t \in [0, \infty)$?

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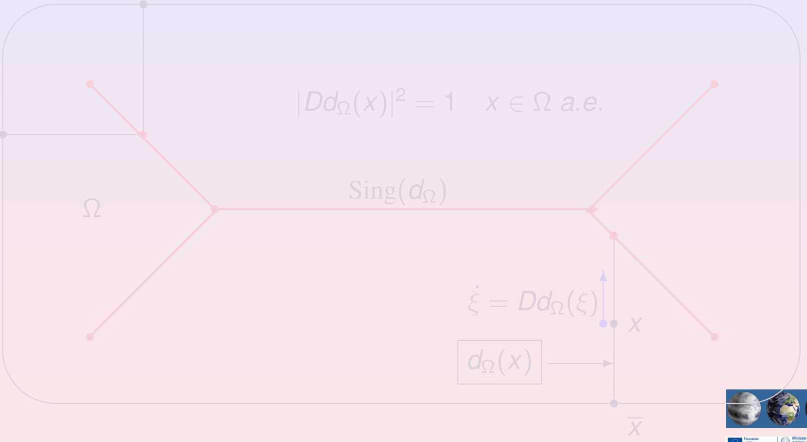
Distance function from $\partial\Omega$

$\Omega \subset \mathbb{R}^n$ bounded open set $\boxed{d_\Omega(x) = \min_{y\in\partial\Omega} |x-y| \quad (x \in \overline{\Omega})}$

- $\mathrm{Sing}(d_\Omega) = \{x \in \Omega \mid \mathrm{proj}_{\partial\Omega}(x) \text{ multivalued}\} \neq \varnothing$



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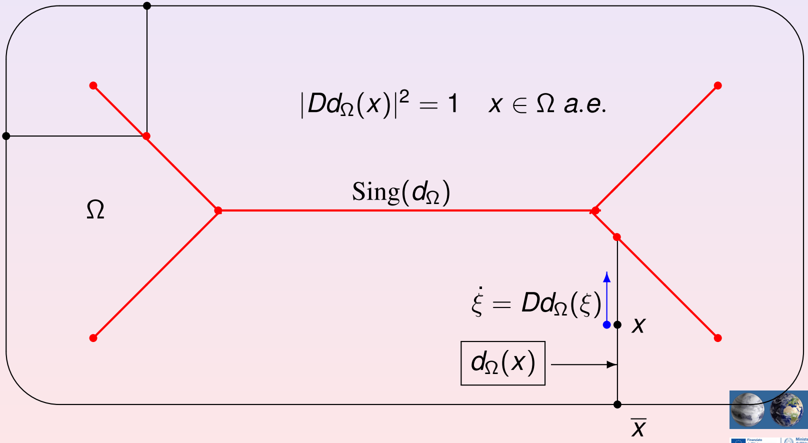
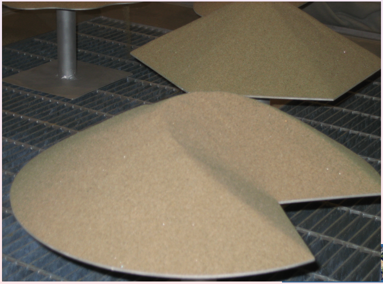


Distance function and sand piles

It is known that *equilibrium configurations* of sand piles on a table $\Omega \subset \mathbb{R}^2$ are described as hypographs of functions proportional to the Euclidean distance from the boundary of Ω

$$d_\Omega(x) = \inf_{y \in \partial\Omega} |x - y| \qquad (x \in \Omega)$$

They naturally exhibit singularities no matter how smooth $\partial\Omega$ is



Global propagation of singularities for d_Ω

For $(t, x_0) \in [0, \infty) \times \Omega$, consider the solution $\mathbf{x}(t, x_0)$ of

$$\begin{cases} \dot{\mathbf{x}}(t) \in D^+ d_\Omega\big(\mathbf{x}(t)\big) & t > 0 \ a.e. \\ \mathbf{x}(0) = x_0 \end{cases}$$

Theorem (Albano – C – Khai T. Nguyen – Sinestrari 2013)

$$x_0 \in \mathrm{Sing}(d_\Omega) \quad \Longrightarrow \quad \mathbf{x}(t) \in \mathrm{Sing}(d_\Omega) \quad \forall t \in [0, \infty)$$

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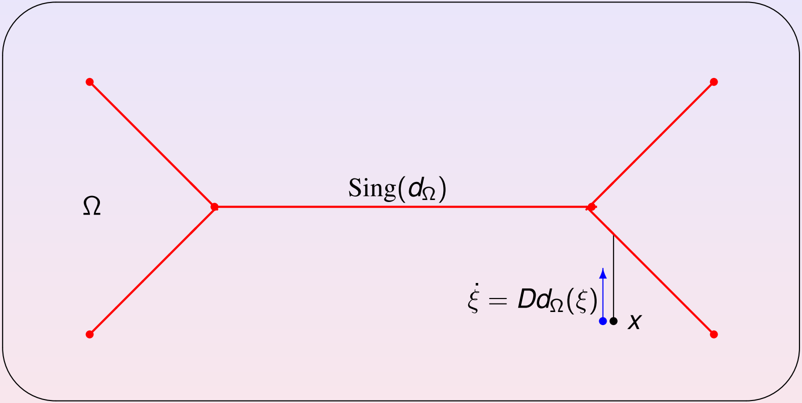

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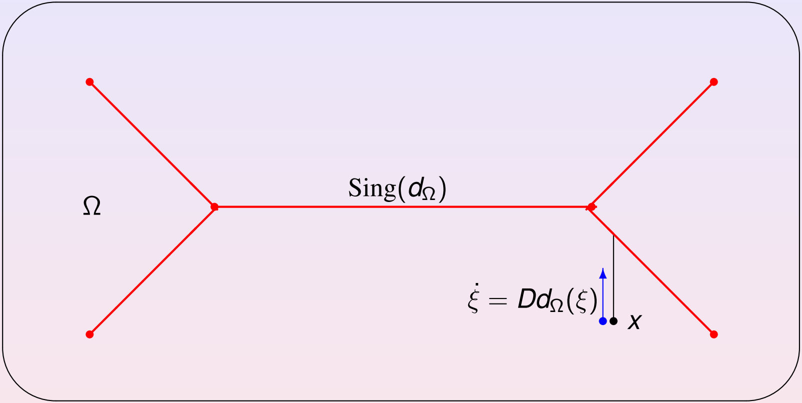

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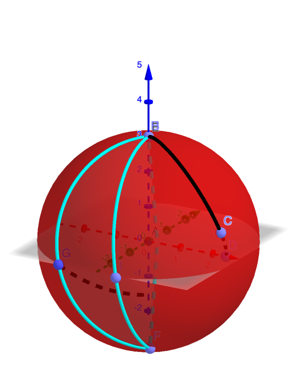
Application to ambiguous locus $\mathcal{NU}(M, g)$

Define sets

- $\mathcal{U}(M, g)$ which consists of all $(x, y) \in M \times M$ for which there exists a unique minimizing g-geodesic $\gamma : [a, b] \to M$ with $x = \gamma(a)$, $y = \gamma(b)$
- $\mathcal{NU}(M, g)$ which consists of all $(x, y) \in M \times M$ that are connected by at least two g-minimizing geodesics



Theorem (C – Cheng – Fathi, 2021)

If M is compact, then $\mathcal{NU}(M, g) \subseteq (M \times M) \setminus \Delta_M$ is a homotopy equivalence and $\mathcal{NU}(M, g)$ is path connected

Application to ambiguous locus $\mathcal{NU}(M, g)$

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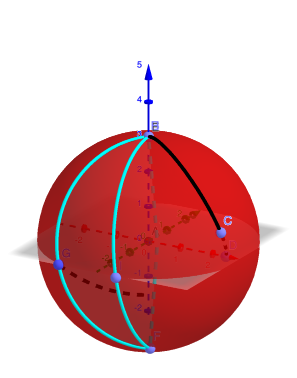

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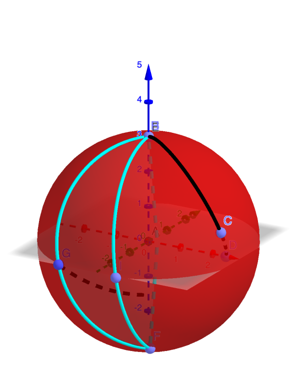

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Closing

- *Global propagation for general Hamiltonians:* true for cut locus; what about genuine singularities?
- *Long time behaviour:* is the critical set an attractor for generalised characteristics?
- *Equations in spaces of measures:* given a solution of the equation $H(x, Du(x)) = 0$ consider the associated *potential energy functional*

$$U[\mu] = \int_{\mathbb{T}^n} u(x)\, d\mu(x) \qquad (\mu \in \mathcal{P}(\mathbb{T}^n))$$

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$$\int_{\mathbb{T}^n} H(x, \partial_\mu U[\mu](x))\, d\mu(x) = 0 \qquad (\mu \in \mathcal{P}(\mathbb{T}^n))$$

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