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Learning-Based Optimization and PDE Control in User-Assignable Finite Time

Miroslav Krstic

with

Cemal Tugrul Yilmaz   Velimir Todorovski

UC San Diego   TU Munich   **(and FAU-Erlangen)**

Outline

- Intro to ES

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- **Prescribed-time (PT)** ES

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- **PT *source*** seeking

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- PT ES under **delays**

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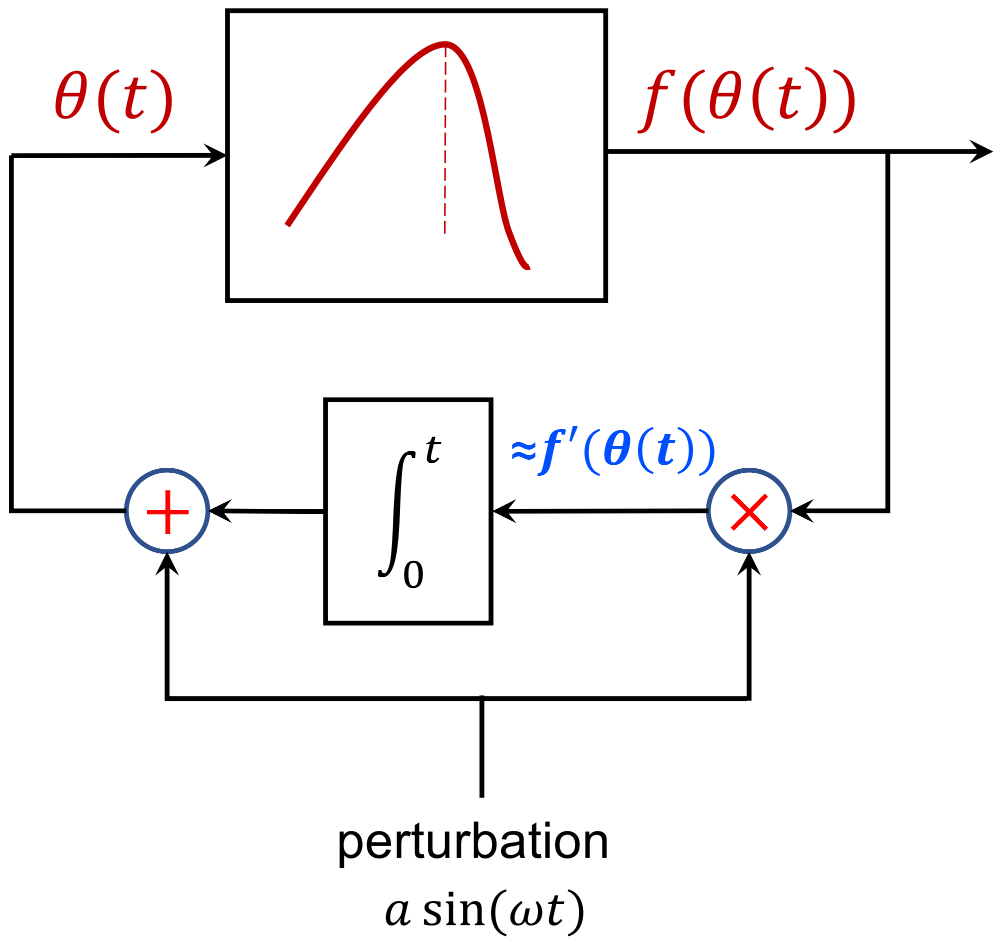
- **Prescribed-time (PT)** ES
- **PT *source*** seeking
- PT ES under **delays**
- PT ES under **PDEs**

Extremum Seeking (1922)



SECTION INDUSTRIELLE

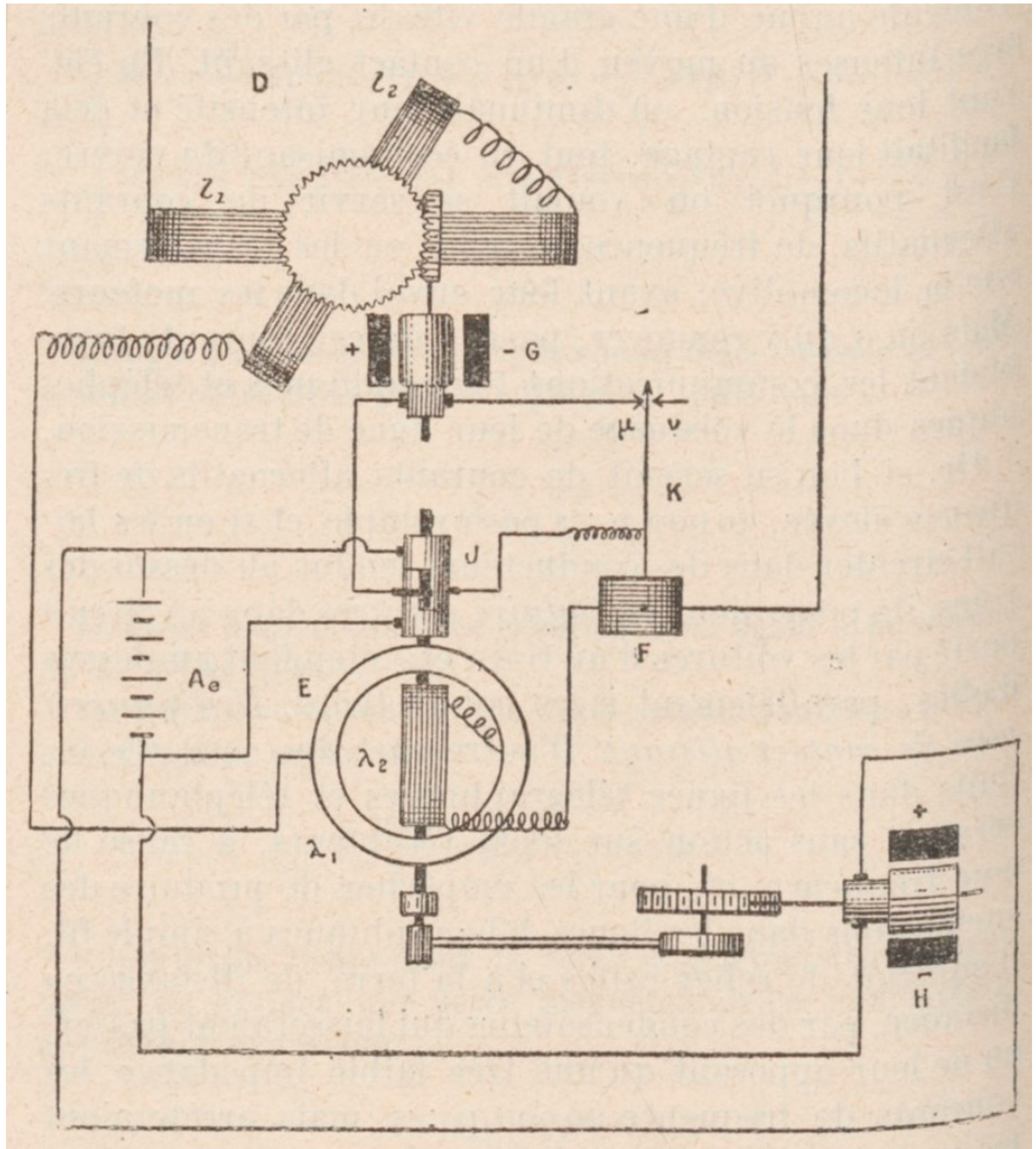
Sur l'électrification des chemins de fer au moyen de courants alternatifs de fréquence élevée

Une légère modification à un dispositif permettant la transformation d'un courant continu en courant alternatif de fréquence élevée, décrit dans la « R. G. E. » du 19 août 1922, t. XII, p. 259-261, a permis à M. Maurice Leblanc d'envisager l'alimentation d'une ligne de transmission d'énergie pour la traction par l'électricité, le récepteur d'énergie électrique n'ayant aucun point de contact avec la ligne de transmission. Outre l'avantage qui consiste à supprimer les contacts glissants, signalons que les courants induits dans les lignes télégraphiques et téléphoniques seraient sans actions sur leurs récepteurs, à cause de leur grande fréquence. La description de ce nouveau dispositif a fait l'objet d'une note présentée à la séance du 17 juillet 1922 de l'Académie des Sciences, que nous reproduisons ci-après.

Introduction. — Il est difficile d'alimenter, un véhicule animé d'une grande vitesse, par des courants très intenses au moyen d'un contact glissant. En élevant leur tension, on diminuait leur intensité et cela facilit...

...cifique égal à 2 et la rigidité électrostatique égale à 400 000 v : cm.

Ces tubes sont interrompus de distance en distance (fig. 1) et chacun d'eux est divisé en tronçons de même



entirely electromechanical!

air-core transformer/capacitor with inductance varied by air-gap

Maurice Leblanc (1857-19**23**)

Inventor & Industrialist

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Contemporary to Robert Bosch and Tesla

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- Electric railway
- Television system
- Induction motors/alternators
- Centrifugal compressors
- Steam jet refrigeration

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Prix Poncelet, French Acad. Sciences **(1913)**

(Boussinesq, Darboux, Cartan, Cosserat, Fréchet, Fredholm, Goursat, Hadamard, Laguerre, Laurent, Lebesgue, Levy, Liouville, Picard, Poincaré)

ES re-awakening

Stability proved in **1997** (MK & Wang)

ES re-awakening

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Torrent of advances: >2,000 papers per year

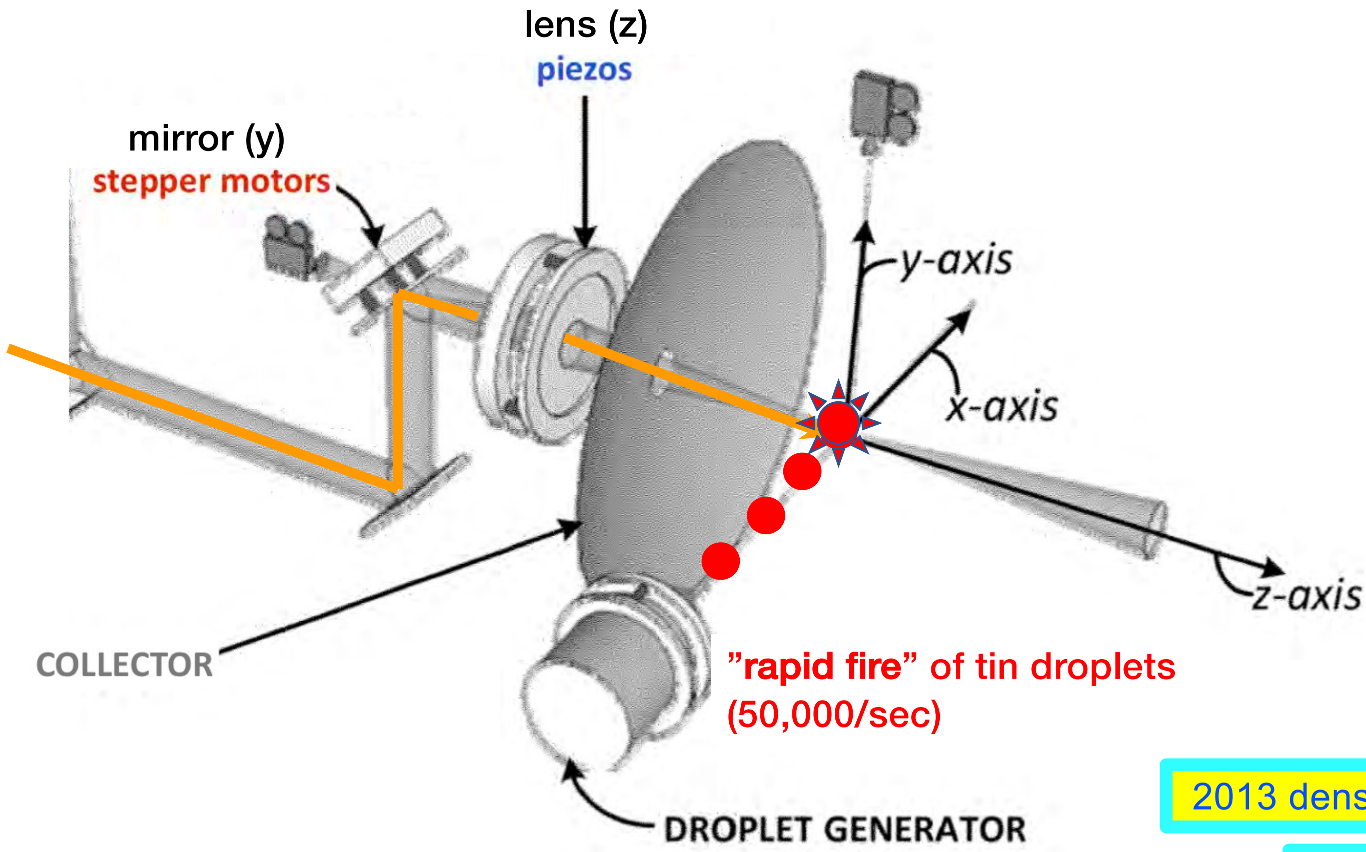
Generators of *Extreme Ultraviolet Light* for photolithography in semiconductor manufacturing

complete wafer scanner **($100M)**

Moore's law (1965)
exceeded physical limit of lasers

IDEA:
evaporation of liquid tin (Sn, 230° C)
and re-condensation generates light of
an order of magnitude smaller wavelength

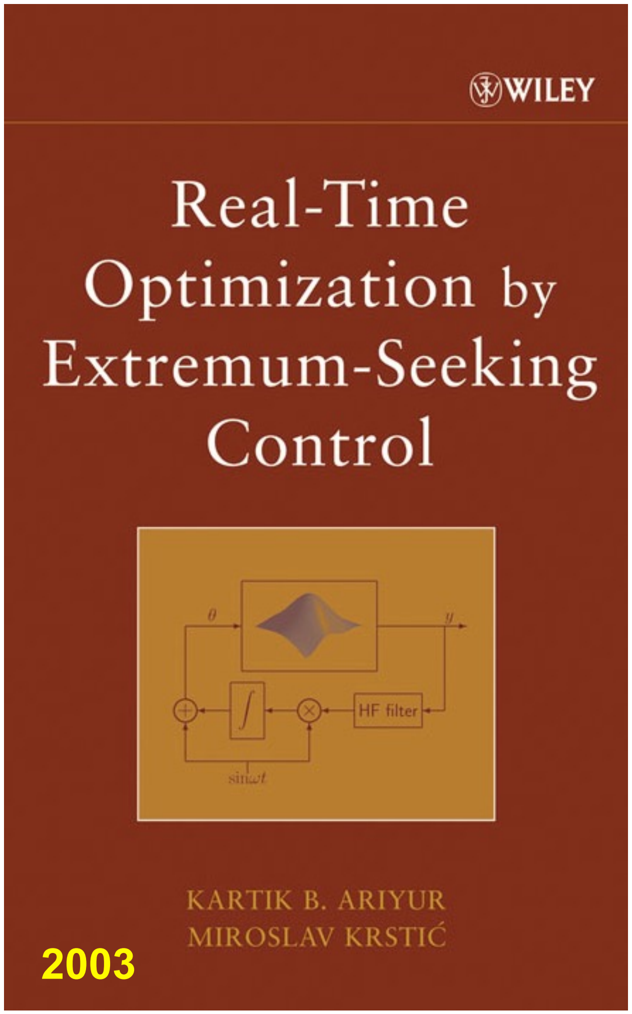
Frihauf et al (2013)

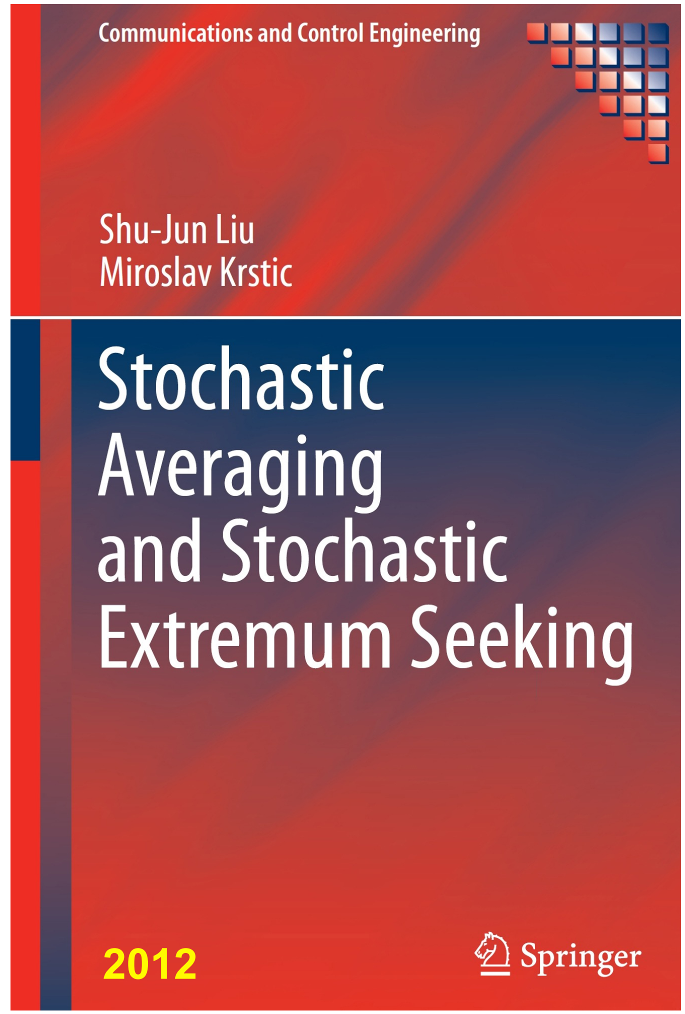


2013 density increase > 200x

May 2021: 2 nm chip!

IBM • Intel • Samsung • TSMC





2022, SIAM: Tiago Roux Oliveira + MK

Extremum Seeking through Delays and PDEs

Basic PT-ES (Static Map)

Gain with unlimited growth

Blowup (at $t = t_0 + T$) function

$$\mu(t) = \frac{1}{1 - \frac{t - t_0}{T}}$$

Gain with unlimited growth

Blowup (at $t = t_0 + T$) function

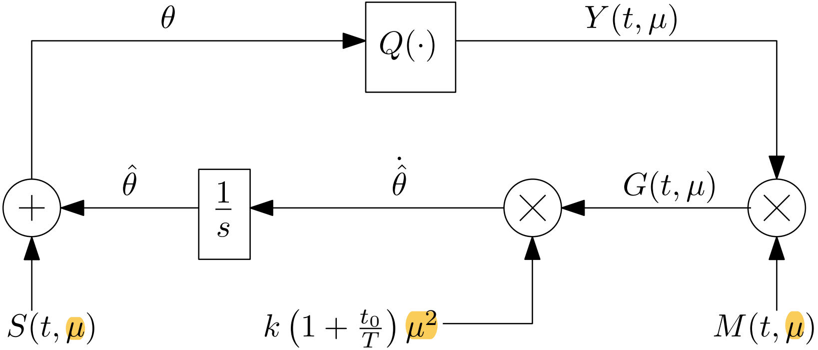
$$\mu(t) = \frac{1}{1 - \frac{t - t_0}{T}}$$

Dynamics of the blowup function

$$\dot{\mu} = \frac{1}{T}\mu^2$$

Basic PT-ES

$$y(t) = y^* + \frac{H}{2}(\theta(t) - \theta^*)^2$$



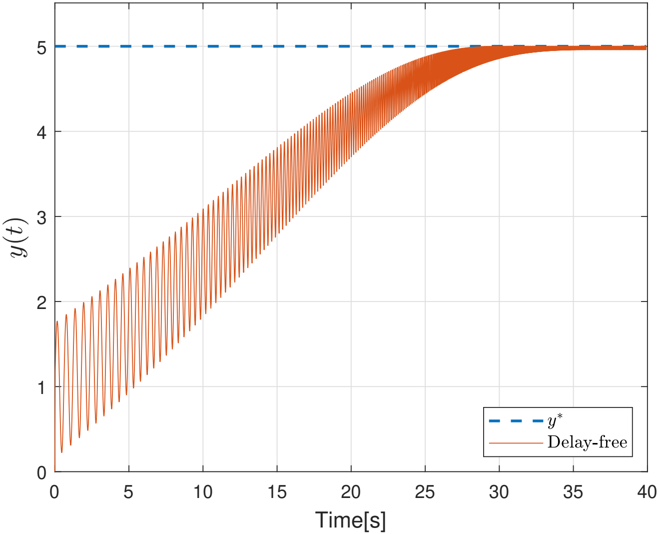
"Chirpy" sinusoid perturbations

Chirp = sine whose frequency grows:

$$S(t,\mu) = a\sin(\omega\mu t)$$
$$M(t,\mu) = \frac{2}{a}\sin(\omega\mu t)$$

Blowup (at $t = t_0 + T$) function

$$\mu(t) = \frac{1}{1 - \frac{t-t_0}{T}}$$



$y(t)$
5
4
3
2
1
0
0
5
10
15
20
25
30
35
40
Time[s]
y^*
Delay-free

Theorem

Consider the PT-ES system with $\tilde{\theta} = \hat{\theta} - \theta^*$ and the dilation/contraction transforms

$$\tau = t\mu(t), \quad t \in [t_0, t_0 + T)$$
$$t = (t_0 + T)\frac{\tau}{T + \tau}, \quad \tau \in [t_0, \infty)$$

Theorem

Consider the PT-ES system with $\tilde{\theta} = \hat{\theta} - \theta^*$ *and the dilation/contraction transforms*

$$\tau = t\mu(t), \qquad t \in [t_0, t_0 + T)$$
$$t = (t_0 + T)\frac{\tau}{T + \tau}, \qquad \tau \in [t_0, \infty)$$

Then $\exists \omega^* > 0$ *s.t.* $\forall \omega > \omega^*$, *the error* $\tilde{\theta}^{\infty}(\tau) = \hat{\theta}\left(\frac{\tau(T+t_0)}{T+\tau}\right) - \theta^*$ *has a unique PT stable sol'n in* t*-domain,* $\tilde{\theta}^{\Pi}(t\mu(t))$, *where* $\tilde{\theta}^{\Pi}(\tau)$ *is the unique exp. stable periodic sol'n in* τ *of period* $\Pi = 2\pi/\omega$ *satisfying*

$$|\tilde{\theta}^{\Pi}(\tau)| \leq O(1/\omega), \qquad \forall \tau \geq t_0.$$

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$$|\tilde{\theta}^\Pi(\tau)| \leq \mathcal{O}(1/\omega), \quad \forall \tau \geq t_0.$$

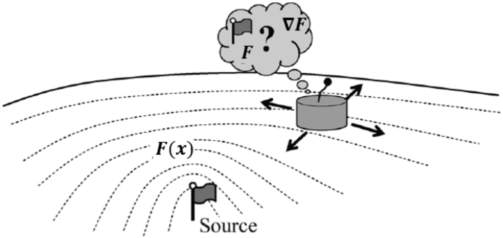
Furthermore,

$$\lim_{t \to t_0 + T} \sup |y(t) - y^*| = \lim_{t \to t_0 + T} \sup |\theta(t) - \theta^*|^2 = \mathcal{O}(a^2 + 1/\omega^2).$$

PT Source Seeking

Velimir Todorovski
TU Munich
(and FAU-Erlangen)

PT Seeking of a (Possibly Repulsive) Source



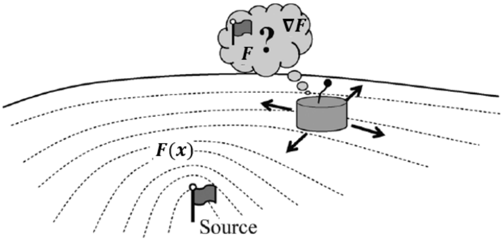
PT Seeking of a (Possibly Repulsive) Source

Unicycle Dynamics:

$$\dot{x} = \underbrace{E\nabla F(x)}_{\text{unknown}} + u_1 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

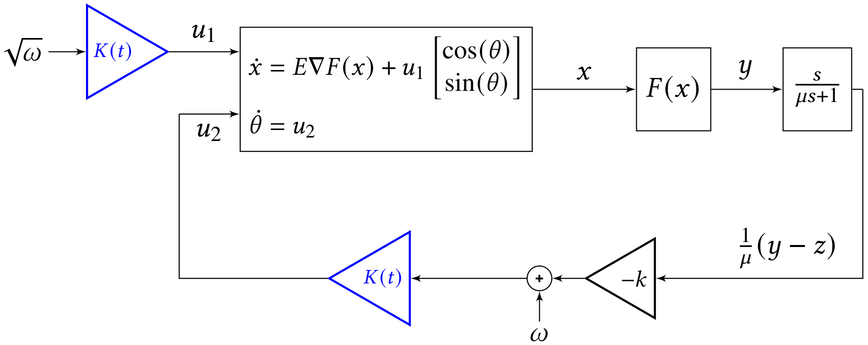
$$\dot{\theta} = u_2$$

$$y = F(x)$$



PT Source Seeker Design

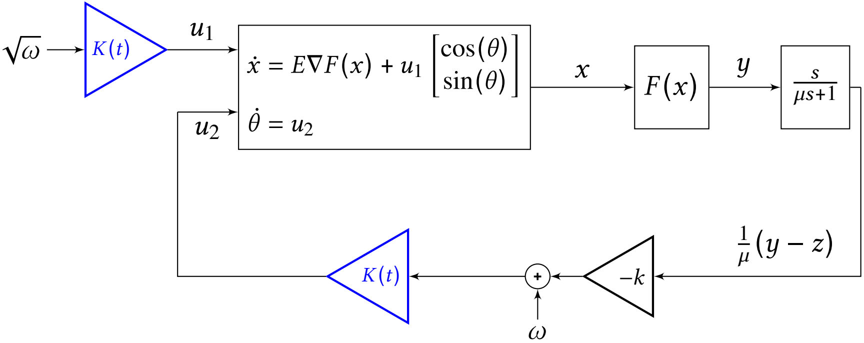
$t \in [t_0, t_0 + T)$



Time-Varying Gain: $K(t) = \left(1 + \frac{t_0}{T}\right) \mu(t)$

PT Source Seeker Design

$t \in [t_0, t_0 + T)$



Time-Varying Gain: $K(t) = \left(1 + \frac{t_0}{T}\right) \mu(t)$

PT version of algorithm by Scheinker, Duerr, Krstic

PT Source Seeker Convergence

Domain $t \in [t_0, t_0 + T)$:

$$\dot{x} = E\nabla F(x) + \frac{1 + \frac{t_0}{T}}{v^2(t - t_0)} \sqrt{\omega} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\dot{\theta} = \frac{1 + \frac{t_0}{T}}{v^2(t - t_0)} \left(\omega - \frac{k}{\mu}(y - z) \right)$$

$$\dot{z} = \frac{1 + \frac{t_0}{T}}{v^2(t - t_0)} \frac{1}{\mu}(y - z)$$

Time Dilation: $\tau = \frac{t}{v(t - t_0)}$

$t \longrightarrow \tau$

$[t_0, t_0 + T) \longrightarrow [t_0, \infty)$

Domain $\tau \in [t_0, \infty)$:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1 + \frac{t_0}{T}}{(1 + \frac{\tau}{T})^2} E\nabla F(x) + \sqrt{\omega} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \omega - \frac{k}{\mu}(y - z)$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{\mu}(y - z)$$

$v(t - t_0) = 1 - \frac{t - t_0}{T}$ where $v(0) = 1$ and $v(T) = 0$

PT Source Seeker Convergence

Domain $t \in [t_0, t_0 + T)$:

$$\dot{x} = E\nabla F(x) + \frac{1 + \frac{t_0}{T}}{\nu^2(t - t_0)} \sqrt{\omega} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\dot{\theta} = \frac{1 + \frac{t_0}{T}}{\nu^2(t - t_0)} \left(\omega - \frac{k}{\mu}(y - z) \right)$$

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$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \omega - \frac{k}{\mu}(y - z)$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{\mu}(y - z)$$

Average Unicycle Position:

$$\left\| \bar{x}(t) - x^* \right\| \leq c \left\| \bar{x}(0) - x^* \right\| e^{|EH|} e^{-\frac{k}{2}\lambda_{\min}(H) \frac{t}{\nu(t-t_0)}}$$

$$\Longrightarrow \bar{x} \longrightarrow x^* \quad \text{as} \quad t \longrightarrow t_0 + T$$

Time Contraction: $t = \dfrac{(t_0 + T)\tau}{\tau + T}$

$t \longleftarrow \tau$

$[t_0, t_0 + T) \longleftarrow [t_0, \infty)$

Average Unicycle Position:*

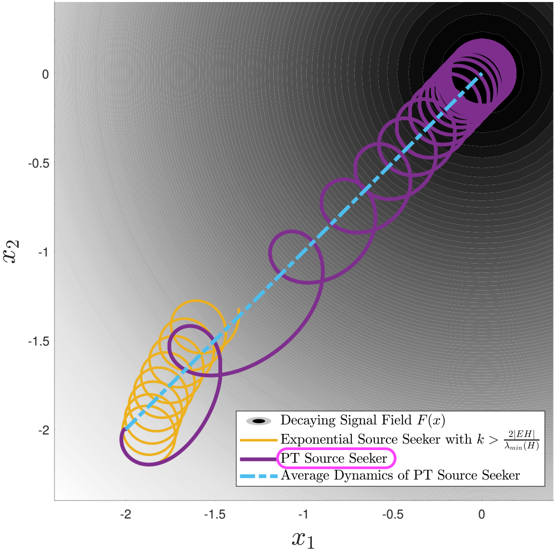
$$\left\| \bar{x}(\tau) - x^* \right\| \leq c \left\| \bar{x}(0) - x^* \right\| e^{|EH|} e^{-\frac{k}{2}\lambda_{\min}(H)\tau}$$

$$\Longrightarrow \bar{x} \longrightarrow x^* \quad \text{as} \quad \tau \longrightarrow \infty$$

*Averaging Results via Sing. perturbed-Lie Bracket Approx.

$\nu(t - t_0) = 1 - \frac{t-t_0}{T}$ where $\nu(0) = 1$ and $\nu(T) = 0$

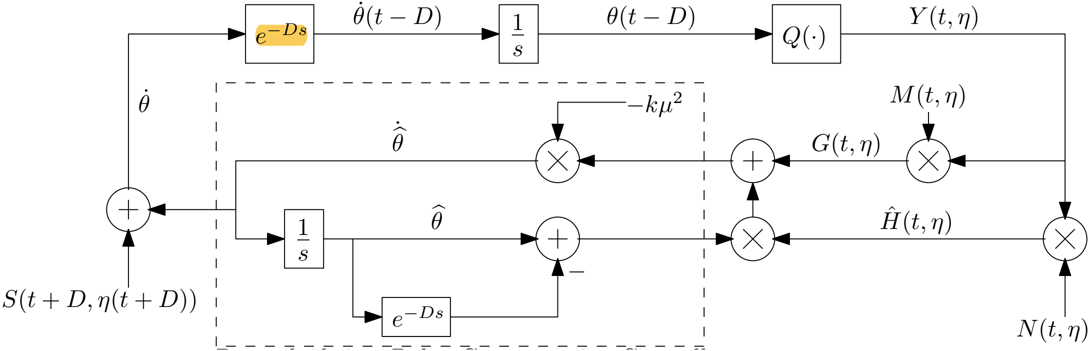
Source Seekers Comparison $t \in [0, 1)$



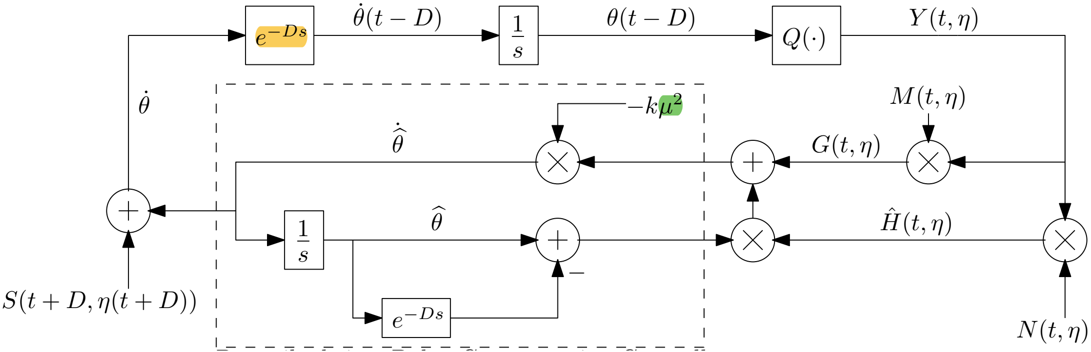
Delay-Compensated PT-ES

(Cemal) Tugrul Yilmaz

Tugrul Yilmaz
U Bogazici
UCSD



e^{-Ds}
$\dot{\theta}(t-D)$
$\frac{1}{s}$
$\theta(t-D)$
$Q(\cdot)$
$Y(t,\eta)$
$\dot{\theta}$
$-k\mu^2$
$\dot{\widehat{\theta}}$
$M(t,\eta)$
$G(t,\eta)$
$\frac{1}{s}$
$\widehat{\theta}$
$\hat{H}(t,\eta)$
$-$
e^{-Ds}
$S(t+D,\eta(t+D))$
$N(t,\eta)$


e^{-Ds}
$\dot{\theta}(t-D)$
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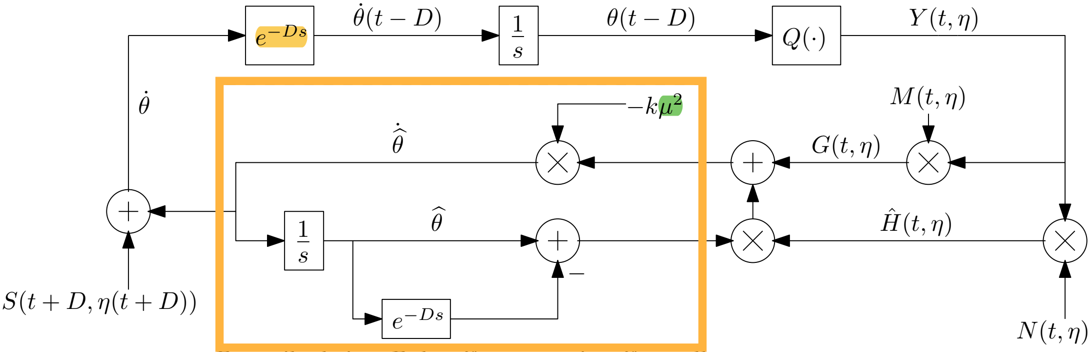
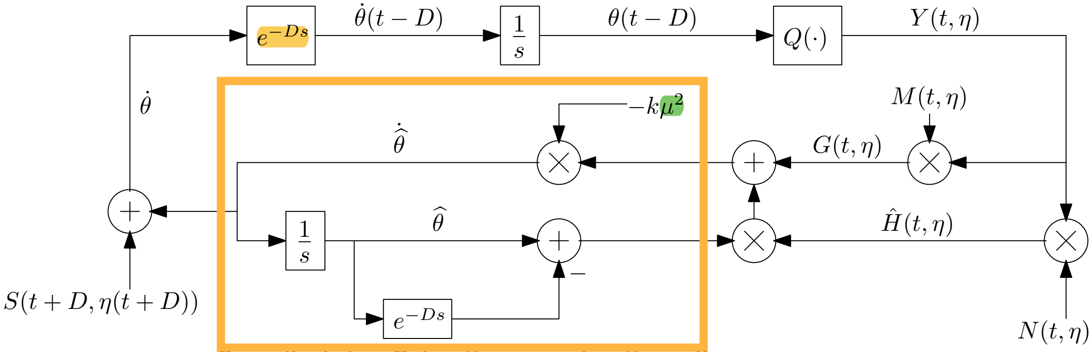
$G(t,\eta)$
$\widehat{\theta}$
$\hat{H}(t,\eta)$
$\frac{1}{s}$
e^{-Ds}
$-$
$S(t+D,\eta(t+D))$
$N(t,\eta)$



Prescribed-time Delay Compensation Controller

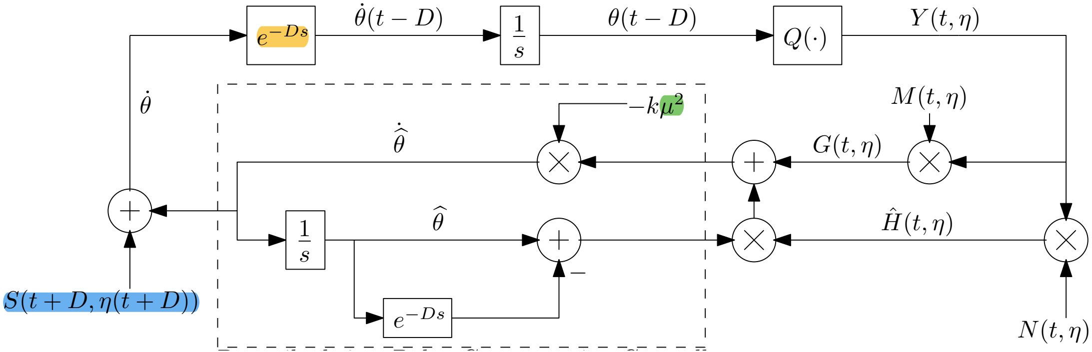
PREDICTOR with distributed delay



Prescribed-time Delay Compensation Controller

PREDICTOR with distributed delay

PT version of algorithm by Tiago Roux Oliveira


e^{-Ds}
$\dot{\theta}(t-D)$
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$-k\mu^2$
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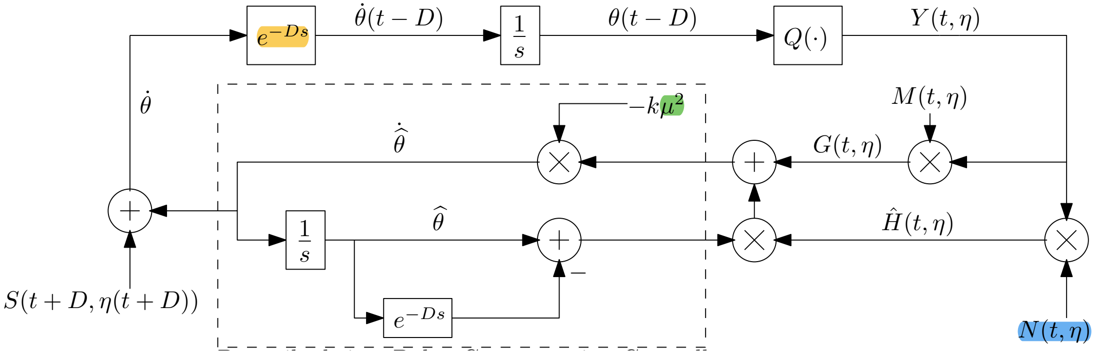
$\dot{\widehat{\theta}}$
$G(t,\eta)$
$\widehat{\theta}$
$\hat{H}(t,\eta)$
$\frac{1}{s}$
$-$
e^{-Ds}
$S(t+D,\eta(t+D))$
$N(t,\eta)$

Time-advanced chirp perturbation

$$S(t+D,\eta(t+D)) = a\omega\left(1+\frac{t_0}{T}\right)\mu(t+D)^2\cos\left(\omega(t+D)\mu(t+D)\right)$$


e^{-Ds}
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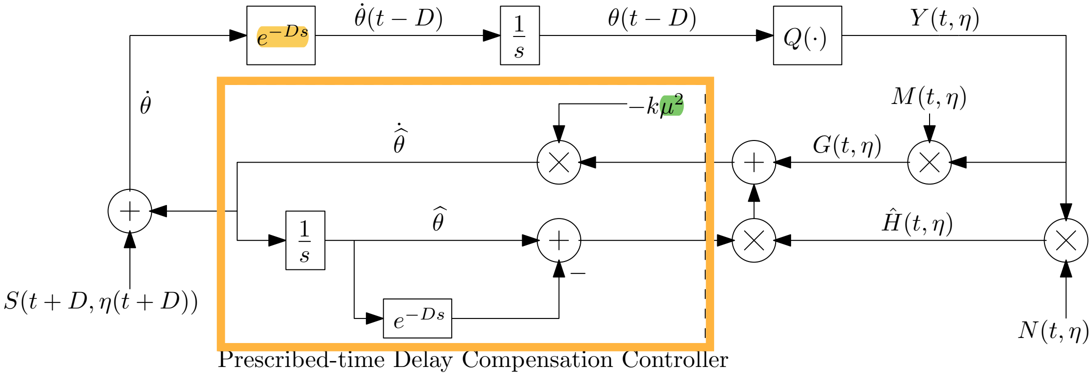
$G(t,\eta)$
$\widehat{\theta}$
$\frac{1}{s}$
$\hat{H}(t,\eta)$
$-$
e^{-Ds}
$S(t+D,\eta(t+D))$
$N(t,\eta)$

Gradient and Hessian estimates

$$G(t,\eta) = M(t,\eta)Y(t,\eta), \qquad M(t,\eta) = \frac{2}{a}\sin(\omega t\mu)$$

$$\hat{H}(t,\eta) = N(t,\eta)Y(t,\eta), \qquad N(t,\eta) = -\frac{8}{a^2}\cos(2\omega t\mu)$$



e^{-Ds}
$\dot{\theta}(t-D)$
$\frac{1}{s}$
$\theta(t-D)$
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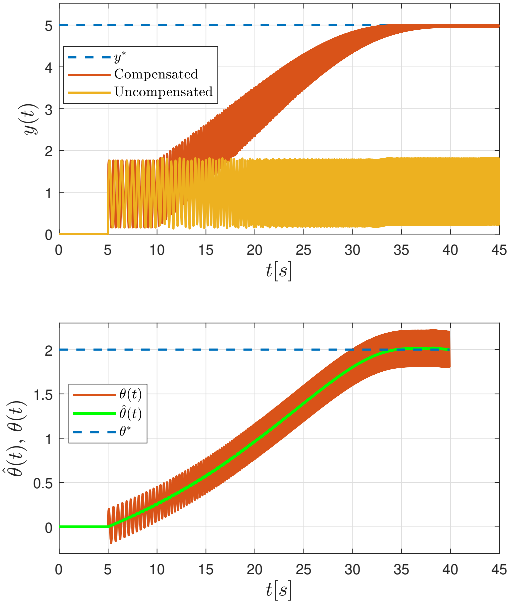
$Y(t,\eta)$
$\dot{\theta}$
$-k\mu^2$
$\dot{\hat{\theta}}$
$M(t,\eta)$
$G(t,\eta)$
$\hat{\theta}$
$\hat{H}(t,\eta)$
$-$
$S(t+D,\eta(t+D))$
$N(t,\eta)$
Prescribed-time Delay Compensation Controller

Delay-compensated PT-ES algorithm

$$\dot{\hat{\theta}}(t) = -k\mu^2(t)\left[\left(\hat{\theta}(t) - \hat{\theta}(t-D)\right)\hat{H}(t,\eta(t)) + G(t,\eta(t))\right]$$

("cancel")


y^*
Compensated
Uncompensated
$y(t)$
$t[s]$
$\theta(t)$
$\hat{\theta}(t)$
θ^*
$\hat{\theta}(t),\ \theta(t)$
$t[s]$

Theorem

For $\bar{T} > D$ arbitrarily close to T, there exists some $\omega^*(\bar{T})$, where $\omega^*(\bar{T}) \to \infty$ as $\bar{T} \to T$, such that $\forall \omega > \omega^*(\bar{T})$, the error system satisfies

$$\limsup_{t \to t_0 + \bar{T} + D} \left(|\vartheta(t)| + ||\bar{u}(\cdot, t)||_{L_2[0,D]} \right) = O(1/\omega),$$

$$\vartheta(t) = \hat{\theta}(t - D) - \theta^*$$

$$\begin{aligned} \vartheta(t) &= \bar{u}(0, t) \\ \partial_t \bar{u}(x, t) &= \partial_x \bar{u}(x, t) \\ \bar{u}(D, t) &= \hat{\theta}(t) - \theta^* \end{aligned}$$

(delay as transport PDE)

Theorem

For $\bar{T} > D$ arbitrarily close to T, there exists some $\omega^*(\bar{T})$, where $\omega^*(\bar{T}) \to \infty$ as $\bar{T} \to T$, such that $\forall \omega > \omega^*(\bar{T})$, the error system satisfies

$$\limsup_{t \to t_0 + \bar{T} + D} \left(|\vartheta(t)| + ||\bar{u}(\cdot, t)||_{L_2[0,D]} \right) = \mathcal{O}(1/\omega),$$

and

$$\limsup_{t \to t_0 + \bar{T}} |\hat{\theta}(t) - \theta^*| = \mathcal{O}(1/\omega).$$

Furthermore,

$$\limsup_{t \to t_0 + \bar{T} + D} |y(t) - y^*| = \limsup_{t \to t_0 + \bar{T}} |\theta(t) - \theta^*|^2 = \mathcal{O}(a^2 + 1/\omega^2).$$

Key step in the proof: **Finite-time weak averaging of inf-dim system with unbounded state**

Evolution eqn

$$\dot{\eta} = \mathscr{A}\eta + J(\omega t, \eta)$$

periodic in t, with μ treated as part of the state η

Key step in the proof: **Finite-time weak averaging of inf-dim system with unbounded state**

Evolution eqn

$$\dot{\eta} = \mathscr{A}\eta + J(\omega t, \eta)$$

periodic in t, with μ treated as part of the state η

Average system

$$\dot{\eta}_{\mathrm{av}} = \mathscr{A}\eta_{\mathrm{av}} + J_{\mathrm{av}}(\eta_{\mathrm{av}})$$

where

$$J_{\mathrm{av}}(\eta) = \lim_{\omega\to\infty} \frac{1}{\omega(\bar{T} - D)} \int_0^{\omega(\bar{T}-D)} J(s, \eta) ds$$

Key step in the proof: **Finite-time weak averaging of inf-dim system with unbounded state**

Weak formulation of evolution eqn (φ = 'test function'):

$$\langle \eta(t), \varphi \rangle = \langle \eta(t_0 + D), \varphi \rangle + \int_{t_0+D}^{t} \langle \eta(s), \mathscr{A}^* \varphi \rangle \, ds + \left\langle \int_{t_0+D}^{t} J_{\text{av}}(\eta(s)) ds, \varphi \right\rangle$$
$$+ \left\langle \int_{t_0+D}^{t} \left[J\left(\omega s, \eta(s)\right) - J_{\text{av}}(\eta(s)) \right] ds, \varphi \right\rangle$$

Key step in the proof: **Finite-time weak averaging of inf-dim system with unbounded state**

Weak formulation of evolution eqn (φ = 'test function'):

$$\langle \eta(t), \varphi \rangle = \langle \eta(t_0 + D), \varphi \rangle + \int_{t_0+D}^{t} \langle \eta(s), \mathscr{A}^* \varphi \rangle \, ds + \left\langle \int_{t_0+D}^{t} J_{\text{av}}(\eta(s)) ds, \varphi \right\rangle$$

$$+ \underbrace{\left\langle \int_{t_0+D}^{t} \left[J\left(\omega s, \eta(s)\right) - J_{\text{av}}(\eta(s)) \right] ds, \varphi \right\rangle}_{\leq \frac{1}{\omega} \delta(\bar{T}) ||\varphi||_{\mathscr{X}} \, = \, O(1/\omega), \quad \forall \omega > \omega^*(\bar{T}) \gg \delta(\bar{T})}$$

Key step in the proof: **Finite-time weak averaging of inf-dim system with unbounded state**

Weak formulation of evolution eqn (φ = 'test function'):

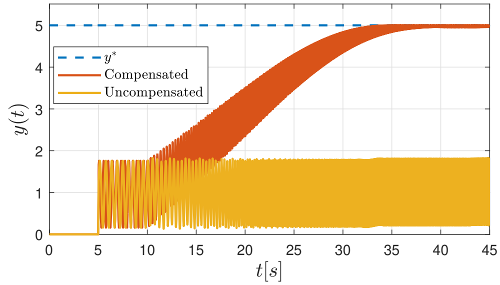
$$\langle \eta(t), \varphi \rangle = \langle \eta(t_0 + D), \varphi \rangle + \int_{t_0+D}^{t} \langle \eta(s), \mathscr{A}^* \varphi \rangle \, ds + \left\langle \int_{t_0+D}^{t} J_{\mathrm{av}}(\eta(s)) ds, \varphi \right\rangle$$

$$+ \underbrace{\left\langle \int_{t_0+D}^{t} \left[J(\omega s, \eta(s)) - J_{\mathrm{av}}(\eta(s)) \right] ds, \varphi \right\rangle}_{\leq \frac{1}{\omega}\delta(\bar{T}) ||\varphi||_{\mathscr{X}} = O(1/\omega), \quad \forall \omega > \omega^*(\bar{T}) \gg \delta(\bar{T})}$$

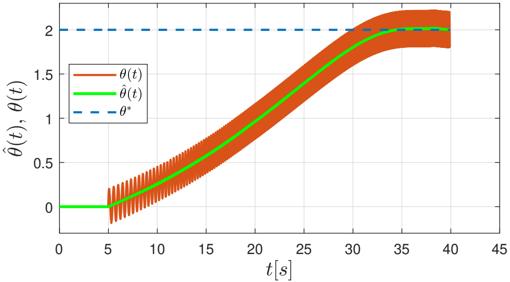
Approximation error on finite interval:

$$||\eta(t) - \eta_{\mathrm{av}}(t)||_{\mathscr{X}} = O(1/\omega), \qquad \forall t \in [t_0 + D, t_0 + \bar{T}]$$

for all $\omega > \omega^*(\bar{T})$.



5
4
3
2
1
0
y(t)
0 5 10 15 20 25 30 35 40 45
t[s]
y*
Compensated
Uncompensated



2
1.5
1
0.5
0
θ̂(t), θ(t)
0 5 10 15 20 25 30 35 40 45
t[s]
θ(t)
θ̂(t)
θ*

PT-ES for Diffusion PDEs

PT-ES for Diffusion PDEs

Diffusion PDE with a (Neumann-driven) integrator at its output

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Theta(t) &= \partial_x \alpha(0,t) \\ \partial_t \alpha(x,t) &= \partial_{xx}\alpha(x,t) \\ \alpha(0,t) &= 0 \\ \partial_x \alpha(D,t) &= \dot{\theta}(t) \end{aligned}$$

(as in Stefan PDE-ODE model of phase change)

Motion planning for perturbation signal —*propagate* chirp thru heat PDE

$$
\begin{aligned}
S(t, \eta(t)) &= \partial_x \beta(D, t) \\
\partial_t \beta(x, t) &= \partial_{xx} \beta(x, t) \\
\beta(0, t) &= 0 \\
\partial_x \beta(0, t) &= a\omega \cos\left(\omega t \mu(t)\right) \left(1 + \frac{t_0}{T}\right) \mu^2(t)
\end{aligned}
$$

Motion planning for perturbation signal —*propagate* chirp thru heat PDE

$$S(t,\eta(t)) = \partial_x \beta(D,t)$$
$$\partial_t \beta(x,t) = \partial_{xx}\beta(x,t)$$
$$\beta(0,t) = 0$$
$$\partial_x \beta(0,t) = a\omega \cos(\omega t \mu(t)) \left(1 + \frac{t_0}{T}\right) \mu^2(t)$$

Explicit solution

$$S(t,\eta) = \mathrm{Re}\left\{ \sum_{k=0}^{\infty} a\omega k_0 \frac{1}{T^k} \mu^{k+2} k! e^{j\omega t \mu} L_k^{(1)}\left(-(Tj\omega k_0)\mu\right) \frac{D^{2k}}{(2k)!} \right\}$$

L_k = Laguerre polynomials

Heat-compensated PT-ES algorithm

$$\dot{\hat{\theta}} = \hat{H}(t,\eta)Q(\eta) + G(t,\eta)P(\eta) + R(\eta)$$

where

$$Q(\eta) = \int_0^D \partial_x q_c(D,y,\eta)(\alpha(y,t) - \beta(y,t))dy$$

$$P(\eta) = \partial_x \gamma_c(D,\eta)$$

$$R(\eta) = q_r(D,D,\eta)(\alpha(D,t) - \beta(D,t)) + \int_0^D \partial_x q_r(D,y,\eta)(\alpha(y,t) - \beta(y,t))dy$$

Kernel q_r

PDE

$$\begin{aligned} \partial_t q_r(x,y,\eta) &= \partial_{xx} q_r(x,y,\eta) - \partial_{yy} q_r(x,y,\eta) - \mu_0 \mu^2 q_r(x,y,\eta) \\ q_r(x,0,\eta) &= 0 \\ q_r(x,x,\eta) &= -\frac{x}{2}\mu_0\mu^2 \end{aligned}$$

Explicit

$$q_r(x,y,\eta) = -y\mu_0\mu^2 \mathrm{e}^{\frac{\mu(x^2-y^2)}{4T}} \frac{I_1\left(\sqrt{\mu_0\mu^2(x^2-y^2)}\right)}{\sqrt{\mu_0\mu^2(x^2-y^2)}}$$

I_1 = modified Bessel function

Kernel q_r

PDE

$$\partial_t q_r(x,y,\eta) = \partial_{xx} q_r(x,y,\eta) - \partial_{yy} q_r(x,y,\eta) - \mu_0 \mu^2 q_r(x,y,\eta)$$
$$q_r(x,0,\eta) = 0$$
$$q_r(x,x,\eta) = -\frac{x}{2}\mu_0 \mu^2$$

Explicit

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I_1 = modified Bessel function

Growth bound

$$|q_r(\cdot,\cdot,\eta)| \leq e^{\left(\frac{D^2}{4T} + C_6\sqrt{\mu_0}\right)\mu}$$

Kernel γ_c

PDE

$$\begin{aligned} \partial_t \gamma_c(x,\eta) &= \partial_{xx}\gamma_c(x,\eta) - \mu_0 \mu^2 \gamma_c(x,\eta) \\ \gamma_c(0,\eta) &= 0 \\ \partial_x \gamma_c(0,\eta) &= -k\mu^2, \quad k < 0 \end{aligned}$$

Explicit

$$\gamma_c(x,\eta) = -k\mu^2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \frac{n!}{T^n} \mu^n L_n^{(1)}(-(T\mu_0)\mu)$$

Laguerre polynomials

Kernel γ_c

PDE

$$\partial_t \gamma_c(x,\eta) = \partial_{xx}\gamma_c(x,\eta) - \mu_0 \mu^2 \gamma_c(x,\eta)$$
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Explicit

$$\gamma_c(x,\eta) = -k\mu^2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \frac{n!}{T^n} \mu^n L_n^{(1)}(-(T\mu_0)\mu)$$

Growth bound

$$|\gamma_c(\cdot,\eta)| \le C_1 e^{C_2 \mu}$$

where $C_1 := \frac{|k|}{\mu_0 D}$, $C_2 := \frac{D^2}{T} + D\sqrt{5\mu_0}$.

Kernel q_c

PDE

$$\partial_t q_c(x,y,\eta) = \partial_{xx} q_c(x,y,\eta) - \partial_{yy} q_c(x,y,\eta) - \mu_0 \mu^2 q_c(x,y,\eta)$$
$$q_c(x,0,\eta) = \gamma_c(x,\eta)$$
$$q_c(x,x,\eta) = 0$$

Successive approx. $\sum_{n=0}^{\infty} Q_c^n(\xi,\delta,\eta) = q_c(\xi+\delta,\xi-\delta,\eta)$

$$Q_c^0(\xi,\delta,\eta) = \gamma_c(2\delta,\eta), \quad Q_c^n(\xi,\delta,\eta) = \int_\delta^\xi \int_0^\delta \left(\partial_t Q_c^{n-1}(\sigma,\beta,\eta) + \mu_0\mu^2 Q_c^{n-1}(\sigma,\beta,\eta)\right) d\beta d\sigma$$

Kernel q_c

PDE

$$\begin{aligned} \partial_t q_c(x,y,\eta) &= \partial_{xx} q_c(x,y,\eta) - \partial_{yy} q_c(x,y,\eta) - \mu_0 \mu^2 q_c(x,y,\eta) \\ q_c(x,0,\eta) &= \gamma_c(x,\eta) \\ q_c(x,x,\eta) &= 0 \end{aligned}$$

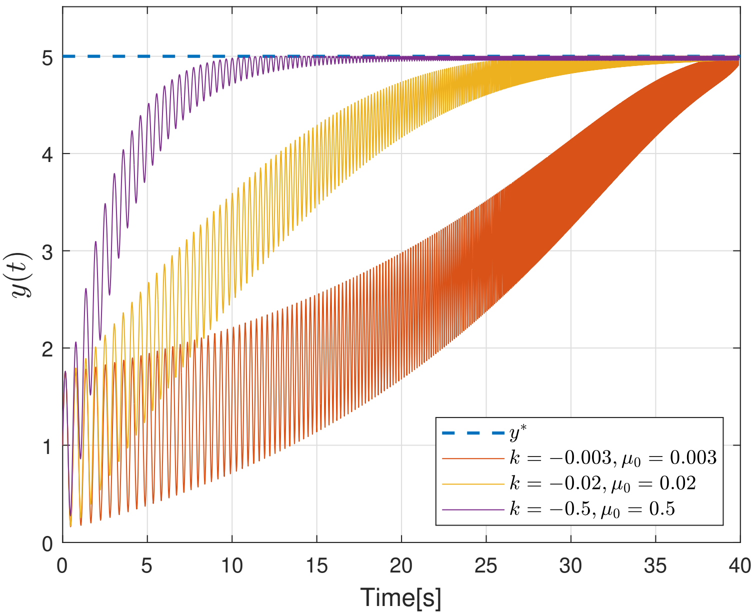
Successive approx. $\sum_{n=0}^{\infty} Q_c^n(\xi,\delta,\eta) = q_c(\xi+\delta,\xi-\delta,\eta)$

$$Q_c^0(\xi,\delta,\eta) = \gamma_c(2\delta,\eta), \quad Q_c^n(\xi,\delta,\eta) = \int_\delta^\xi \int_0^\delta \Big(\partial_t Q_c^{n-1}(\sigma,\beta,\eta) + \mu_0\mu^2 Q_c^{n-1}(\sigma,\beta,\eta)\Big) d\beta d\sigma$$

Growth bound

$$|q_c(\cdot,\cdot,\eta)| \leq C_4 e^{C_5 \mu}$$

where $C_4 := 2C_1$, $C_5 := \sqrt{5(C_2+\mu_0 T)D^2/(4T)} + C_2 + D^2/(4T)$



5
4
3
2
1
0
$y(t)$
0
5
10
15
20
25
30
35
40
Time[s]
y^*
$k = -0.003, \mu_0 = 0.003$
$k = -0.02, \mu_0 = 0.02$
$k = -0.5, \mu_0 = 0.5$

Theorem

For $\bar{T} > 0$ arbitrarily close to T, there exists some $\omega^*(\bar{T})$, where $\omega^*(\bar{T}) \to \infty$ as $\bar{T} \to T$, such that $\forall \omega > \omega^*(\bar{T})$, the error system satisfies

$$\limsup_{t \to t_0 + \bar{T}} \left(|\vartheta(t)| + ||\bar{u}(\cdot, t)||_{L_2[0,D]} + ||\partial_x \bar{u}(\cdot, t)||_{L_2[0,D]} + |\hat{\theta}(t) - \Theta^*| \right) = O(1/\omega).$$

Theorem

For $\bar{T} > 0$ arbitrarily close to T, there exists some $\omega^*(\bar{T})$, where $\omega^*(\bar{T}) \to \infty$ as $\bar{T} \to T$, such that $\forall \omega > \omega^*(\bar{T})$, the error system satisfies

$$\limsup_{t \to t_0 + \bar{T}} \left(|\vartheta(t)| + ||\bar{u}(\cdot, t)||_{L_2[0,D]} + ||\partial_x \bar{u}(\cdot, t)||_{L_2[0,D]} + |\hat{\theta}(t) - \Theta^*| \right) = \mathcal{O}(1/\omega).$$

Furthermore,

$$\limsup_{t \to t_0 + \bar{T}} |y(t) - y^*| = \limsup_{t \to t_0 + \bar{T}} |\Theta(t) - \Theta^*|^2 = \mathcal{O}(a^2 + 1/\omega^2).$$

Proof Step 1: **Averaging**

$$\begin{aligned}
\dot{\vartheta}_{\text{av}}(t) &= \partial_x u_{\text{av}}(0,t) \\
\partial_t u_{\text{av}}(x,t) &= \partial_{xx} u_{\text{av}}(x,t) \\
u_{\text{av}}(0,t) &= 0, \\
\partial_x u_{\text{av}}(D,t) &= q(D,D,\eta)u_{\text{av}}(D,t) + \int_0^D \partial_x q(D,y,\eta)u_{\text{av}}(y,t)dy + \partial_x \gamma(D,\eta)\vartheta_{\text{av}}
\end{aligned}$$

Proof Step 2: **Backstepping transformation**

$$w(x,t) = u_{\mathrm{av}}(x,t) - \int_0^x q(x,y,\eta)u_{\mathrm{av}}(y,t)dy - \gamma(x,\eta)\vartheta_{\mathrm{av}}$$

into the target system

$$\begin{aligned}
\dot{\vartheta}_{\mathrm{av}}(t) &= -kH\mu^2(t)\vartheta_{\mathrm{av}}(t) + \partial_x w(0,t)\\
\partial_t w(x,t) &= \partial_{xx} w(x,t) - \mu_0\mu^2(t)w(x,t)\\
w(0,t) &= 0\\
\partial_x w(D,t) &= 0
\end{aligned}$$

Proof Step 2: **Backstepping transformation**

$$w(x,t) = u_{\rm av}(x,t) - \int_0^x q(x,y,\eta)u_{\rm av}(y,t)dy - \gamma(x,\eta)\vartheta_{\rm av}$$

into the target system

$$\dot{\vartheta}_{\rm av}(t) = -kH\mu^2(t)\vartheta_{\rm av}(t) + \partial_x w(0,t)$$

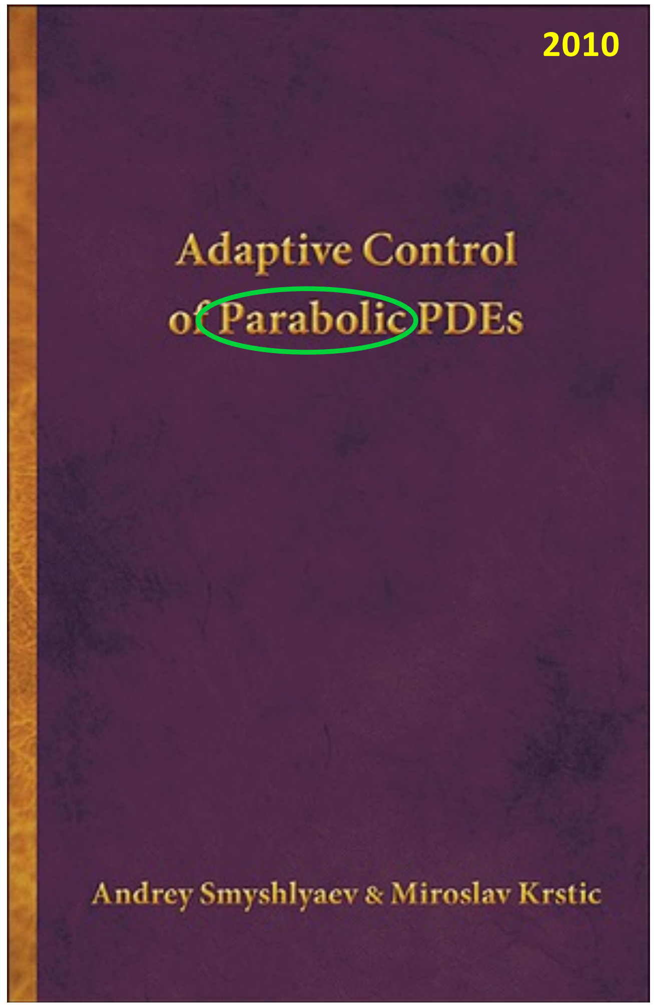
$$\partial_t w(x,t) = \partial_{xx} w(x,t) - \mu_0\mu^2(t) w(x,t)$$

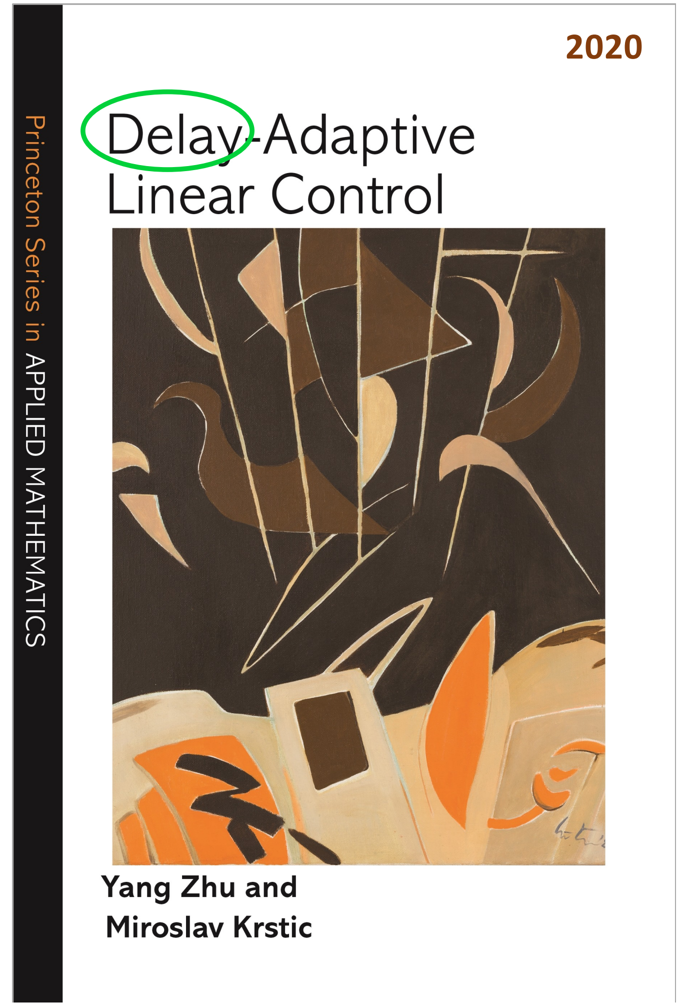
$$w(0,t) = 0$$

$$\partial_x w(D,t) = 0$$

partly inspired by Coron's use of "PDE backstepping" to prove null controllability of heat eqn in time T

Backstepping: from ODEs to PDEs to delays





Proof Step 3: **Stability of target system**

Lyapunov functional

$$\Upsilon(t) = \frac{\vartheta_{\rm av}^2(t)}{2} + \frac{1}{2}||w(\cdot,t)||^2_{L_2[0,D]} + \frac{1}{2}||\partial_x w(\cdot,t)||^2_{L_2[0,D]}$$

Proof Step 3: **Stability of target system**

Lyapunov functional

$$\Upsilon(t) = \frac{\vartheta_{\mathrm{av}}^2(t)}{2} + \frac{1}{2}\|w(\cdot,t)\|_{L_2[0,D]}^2 + \frac{1}{2}\|\partial_x w(\cdot,t)\|_{L_2[0,D]}^2$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(t) \leq & -\left(kH\mu^2(t) - \frac{1}{2}\right)\vartheta_{\mathrm{av}}^2(t) - \mu_0\mu^2(t)\|w(\cdot,t)\|_{L_2[0,D]}^2 \\ & -\left(\mu_0\mu^2(t) + \frac{1}{2}\right)\|\partial_x w(\cdot,t)\|_{L_2[0,D]}^2 - \frac{1}{2}\|\partial_{xx} w(\cdot,t)\|_{L_2[0,D]}^2\end{aligned}$$

Proof Step 3: **Stability of target system**

Lyapunov functional

$$\Upsilon(t) = \frac{\vartheta_{\mathrm{av}}^2(t)}{2} + \frac{1}{2}||w(\cdot,t)||^2_{L_2[0,D]} + \frac{1}{2}||\partial_x w(\cdot,t)||^2_{L_2[0,D]}$$

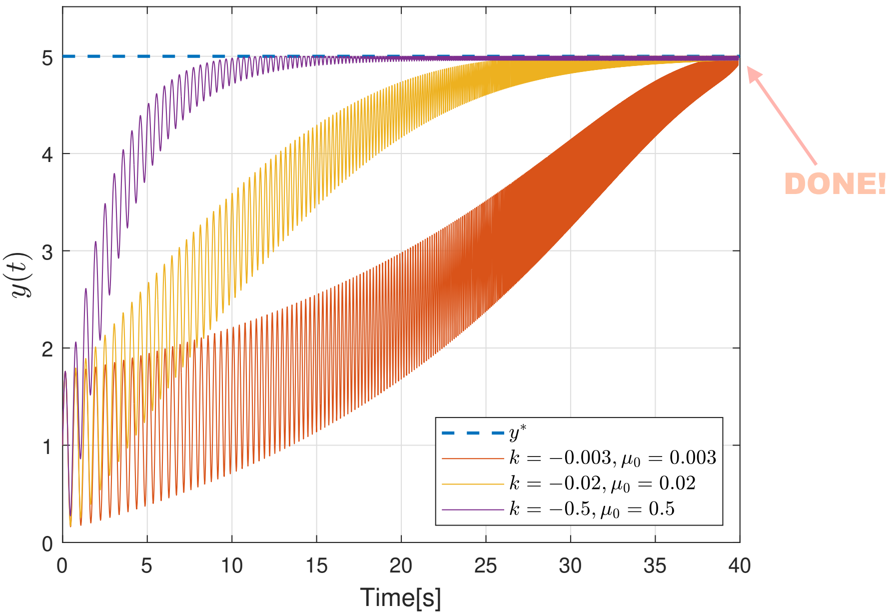
$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(t) \leq -\left(kH\mu^2(t) - \frac{1}{2}\right)\vartheta_{\mathrm{av}}^2(t) - \mu_0\mu^2(t)||w(\cdot,t)||^2_{L_2[0,D]}$$
$$-\left(\mu_0\mu^2(t) + \frac{1}{2}\right)||\partial_x w(\cdot,t)||^2_{L_2[0,D]} - \frac{1}{2}||\partial_{xx} w(\cdot,t)||^2_{L_2[0,D]}$$

Choosing

$$k = \frac{2\hat{k}+1}{2\bar{H}}, \qquad \mu_0 = \frac{2\hat{\mu}_0+1}{2} > \frac{D^2}{T^2}$$

with $\hat{\mu}_0, \hat{k} > 0$, we get

$$\Upsilon(t) \leq \mathrm{e}^{-c_0 T(\mu(t)-1)}\Upsilon(t_0), \qquad c_0 = \min\{\hat{k}, \hat{\mu}_0\}$$



$y(t)$
Time[s]
y^*
$k = -0.003, \mu_0 = 0.003$
$k = -0.02, \mu_0 = 0.02$
$k = -0.5, \mu_0 = 0.5$
DONE!

Thank you for your attention

Questions?