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RIGIDITY OF CURVATURE BOUNDS OF QUOTIENT SPACES OF ISOMETRIC ACTIONS

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Introduction

The Ricci curvature is unquestionably a subject of main importance in geometry. Roughly speaking, it measures how shape is deformed along geodesics. And therefore, it is of great significance in physics. Take for example **The lazy gas experiment**, in page 459 of [?], which relates the concavity of the entropy of a gas being transported in a given space and the curvature of this space; here, if the density of the gas is given by the function ρ, its entropy is $-\int \rho \log \rho$.
Let M be a Riemannian manifold. In this setting, the opposite of entropy is the well known H-**Boltzmann functional**, by

$$H(\mu) = \int_M \rho \log \rho d\,\mathrm{vol}, \quad \mu = \rho\,\mathrm{vol}\,.$$

And in fact, later results in Optimal Transport Theory have shown that convexity properties of this functional guarantee necessary and sufficient conditions to lower bounds on the Ricci curvature of M.
Based on that, we introduce a new H-functional on the total space of an isometric Lie group action to investigate the Ricci curvatures of quotient spaces of Lie groups isometric actions.
More specifically we prove the following theorem.
Theorem 1. Let $G \curvearrowright M$ be an isometric group action with M being a complete Riemannian manifold equipped with geodesic distance d and $\dim M = N$. Therefore, for any $K \in \mathbb{R}$, H is locally $K\Lambda_N$-displacement convex if, and only if, the Ricci curvatures of the quotient space M/G on the principal strata are bounded below by K.

Wasserstein Spaces and Displacement Convexit

First of all we must introduce the domain of our soon to be H-operator.

Definition. Let (X,d) be a polish space. We define the **Wasserstein distance** of order 2 as $W_2 : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$W_2(\mu,\nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} d(x_1,x_2)^2 d\pi(x_1,x_2) \right)^{1/2}.$$

Definition. The **Wasserstein Space** of order 2 is, fixed $x_0 \in X$,

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) : W_2(\mu,\delta_{x_0}) = \int_{X\times X} d(x,x_0)^2 d\mu(x) < +\infty \right\}.$$

Let Γ be the set of constant speed geodesics on M. The geodesics on $P_2(M)$ are of the form $\mu_t := (e_t)_*\Pi$; with Π being a probability measure on Γ and e_t being the evaluation maps $\Gamma \ni \gamma \mapsto \gamma_t \in M$.
These spaces are geodesically connected.

Definition. Let (M,g) be a Riemannian manifold and $x \mapsto \Lambda(x,v)$ a continuous function with range on the quadratic forms on TM. A function $F : \mathcal{P}_2^{ac}(M) \to \mathbb{R} \cup \{+\infty\}$ is said to be Λ-**displacement convex** if for any constant speed minimizing geodesic $(\mu_t)_{0\le t\le 1}$, with an associated Hamilton-Jacobi equation, for any $t \in [0,1]$,

$$F(\mu_t) \le (1-t)F(\mu_0) + tF(\mu_1) - \int_0^1 \Lambda(\mu_s, \tilde{\nabla}\psi_s) G(s,t) ds; \qquad (1)$$

It is assumed that $\Lambda(\mu_s, \tilde{\nabla}\psi_s)G(s,t)$ is bounded below by an integrable function of $s \in [0,1]$.

Disintegration of Absolutely Continuous Measures

As the natural projection $\pi : M \to M/G$ is continuous it is a Borel map. Then we may apply the following theorem on disintegration of measures, whose proof can be found at [?].

Theorem. Let X and Y be locally compact and separable metric spaces and $\pi : X \to Y$ be a Borel map. Fix $\mu \in \mathcal{M}_+(X)$ – where $\mathcal{M}_+(X)$ is the set of positive and finite Radon measures on X – and define $\nu = \pi_*\mu \in \mathcal{M}_+(Y)$. Then, there are measures $\mu_y \in \mathcal{M}_+(X)$ such that

- $y \mapsto \mu_y$ is a Borel map and μ_y is a probability on X for ν-almost every $y \in Y$;
- $\mu = \nu \otimes \mu_y$; that is, $\mu(A) = \int_Y \mu_y(A) d\nu(y)$ for every measurable set $A \subset X$;
- μ_y is concentrated on $\pi^{-1}(y)$ for ν-almost every $y \in Y$.

Fixed a normal slice Σ on a principal point of the action and Z a principal isotropy group we have that the volume measure is locally $\mathrm{vol} = \mathrm{vol}_{G/Z} \times \mathrm{vol}_\Sigma$.
For each $x \in \Sigma$, define $\mathrm{vol}_\rho^x := \rho\,\mathrm{vol}_{G/Z} \times \delta_x$ and $\mu_x = \frac{\mathrm{vol}_\rho^x}{\mathrm{vol}_\rho^x\left(\frac{G}{Z}\times\{x\}\right)}$. It is an easy task to prove that μ_x gives a disintegration of μ with respect to the principal orbits of the action. It can be seen as a measure on G/Z that is absolutely continuous to $\mathrm{vol}_0 := \mathrm{vol}_x$ for any $x \in \Sigma$.

Example: S^1 acting on $\mathbb{R}^2$

Consider the action of the group $S^1 = \{e^{i\theta} : \theta \in [0,2\pi]\}$ on the complex plane $\mathbb{C}$ given by the complex product, i.e., $e^{i\theta}$ acts on $z \in \mathbb{C}$ via $e^{i\theta}z$. Writing the complex number in polar form, $z = re^{i\alpha}$ with $r \in [0,+\infty[$ and $\alpha \in [0,2\pi]$, we see that $e^{i\theta}\cdot re^{i\alpha} = re^{i(\theta+\alpha)}$. Thus, this action is given as $(\cos\theta,\sin\theta)\cdot(r\cos\alpha, r\sin\alpha) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$. The orbit of a point $(x,y) \in \mathbb{C}$ is the circle $S^1\cdot(x,y) = \{(r\cos\theta, r\sin\theta) : r = |(x,y)|, \theta \in [0,1]\}$, so the action does not change the radius of the initial point. This proves that the action is indeed under isometries and that the orbits are concentric circles centred at the origin, except for the orbit of the point 0, that is degenerate: $\{0\}$. Furthermore, each orbit intersects the interval $[0,+\infty[$ exactly once. We may identify this segment of non-negative real number with $\mathbb{C}/S^1$. By understanding the geometry of $[0,+\infty[$ and of the orbits, one understands the geometry of the whole complex plane.
Consider $\pi : \mathbb{R}^2 \to [0,+\infty[$ the projection of this action. Then,

$$\pi(r\cos\theta, r\sin\theta) = r.$$

The volume form of $\mathbb{C}$ can be written as

$$d\,\mathrm{vol} = rdr \wedge d\theta.$$

Thus,

$$\mathrm{vol}(A) = \int_A rdr \wedge d\theta.$$

This measure actually induces a measure on the quotient space $[0,+\infty[$ by the pushfoward measure $\pi_*\,\mathrm{vol}$, which is given by

$$\pi_*\,\mathrm{vol}(U) = \mathrm{vol}(S^1\cdot U) = \int_{S^1\cdot U} rdr \wedge d\theta.$$

We may also "split" vol into conditional measures vol_r whose support is contained in the circle of radius r that are defined via

$$d(\mathrm{vol}_r) = d(\delta_r) \wedge d\theta.$$

Those measures can be "glued together" to "build" vol back. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}_+} \mathrm{vol}_r(A) d(\pi_*\,\mathrm{vol})(r) &= \int_{\mathbb{R}_+} \left(\int_A d(\delta_r \times \theta)(z,\theta) \right) d(\pi_*\,\mathrm{vol})(r) \\
&= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^2} \chi_A(z,\theta) d(\delta_r\times\theta)(z,\theta) \right) d(\pi_*\,\mathrm{vol})(r) \\
&= \int_{\mathbb{R}_+} \left(\int_0^1 \chi_A(r,\theta) d\theta(\theta) \right) d(\pi_*\,\mathrm{vol})(r) \\
&= \int_{\mathbb{R}^2} \left(\int_0^1 \chi_A(r,\theta) d\theta(\theta) \right) d\,\mathrm{vol}(r,\tilde{\theta}) \\
&= \int_{\mathbb{R}_+} \int_0^1 \int_0^1 \chi_A(r,\theta) r d\theta dr d\tilde{\theta} \\
&= \int_{\mathbb{R}_+} \int_0^1 \chi_A(r,\theta) r dr d\theta = \mathrm{vol}(A).
\end{aligned}$$

The H-Functional

In order to make sense to the statement of the Theorem 1, we finally define the H-function on the space $\mathcal{P}_2^{ac}(G/Z)$ via

$$H(\mu) := -\int_{G/Z} N(\rho^{1-1/N} - \rho) d\,\mathrm{vol}_0, \quad \mu = \rho\,\mathrm{vol}_0;$$

with $N = \dim M$.
And the quadratic norm:

$$\Lambda_N(\mu, v) := \int_{G/Z} |v(z)|^2 \rho^{1-1/n}(z) d\,\mathrm{vol}_0(z), \quad \mu = \rho\,\mathrm{vol}_0\,.$$

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Selected publications

[1] Gomes, A., Rodrigues S. (2023) **Rigidity of Curvature Bounds of Quotient Spaces Of Isometric Actions.** arXiv preprint arXiv:2310.15332.

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[4] Possobon, Renata; Rodrigues, Christian S. (2022) **Geometric properties of disintegration of measures.** arXiv preprint arXiv:2202.04511.

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