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DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

SHADOWING ON HILBERT SPACES

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Introduction

In this work we extend results about Shadowing Lemma there are known on finite dimensional compact manifolds without border (or $\mathbb{R}^n$), to an infinite dimensional space. In fact, we proved that if $\{\mathcal{T}(t) : t \geq 0\}$ is a Morse-Smale semigroup defined on a Hilbert space with global attractor $\mathcal{A}$, then $\mathcal{T}(1)$ admits the Lipschitz Shadowing on $\mathcal{A}$ and there exists a neighborhood of $\mathcal{A}$ such that $\mathcal{T}(1)$ has the Holder-Shadowing property. Moreover, we proved a new way to obtain continuity of global attractors using pseudo orbits even when we do not have the Shadowing property.

Morse-Smale Semigroups and Shadowing

Definition

Let (M, d) be a metric space and $\mathcal{C}(M)$ be the space of continuous functions from M into M. We say that the family $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}(M)$ is a semigroup if satisfies:

1. $\mathcal{T}(0)x = x, \ \forall x \in M$;
2. $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s), \ \forall t, s \in \mathbb{R}$;
3. The map $\mathbb{R}_+ \times M \ni (t, x) \mapsto \mathcal{T}(t)x$ is continuous.

Definition

Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\}$ be a semigroup on a metric space (M, d). We say that a compact set $\mathcal{A} \subset X$ is a **global attractor** if its invariant, i.e. $\mathcal{T}(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and attracts bounded subsets of M, that is, for each bounded subset B of M

$$\lim_{t \to +\infty} dist_H(\mathcal{T}(t)B, \mathcal{A}) = 0, \tag{1}$$

where $dist_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$ is the Hausdorff semi-distance between $A, B \subset M$.

Definition

Let X be a Banach space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\}$ be a semigroup on X. We say that $x \in X$ is a **non-wandering** point of $\mathcal{T}(\cdot)$ if for any $t_0 \geq 0$ and neighborhood V_x of x there exists $t > t_0$ such that $\mathcal{T}(t)V_x \cap V_x \neq \emptyset$. We will denote the set of non-wandering point of $\mathcal{T}(\cdot)$ by Ω.

Definition (Morse-Smale)

Let X be a Banach space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}^1(X)$ be a semigroup with global attractor $\mathcal{A}$. We say that $\mathcal{T}(\cdot)$ is a **Morse-Smale** semigroup if it satisfies the following conditions:

1. $\mathcal{T}(t)|_{\mathcal{A}}$ is injective for all $t \geq 0$.
2. The Fréchet derivative $D_z\mathcal{T}(t) \in \mathcal{L}(X)$ of $\mathcal{T}(t)$ at z is an isomorphism onto its image for all $z \in \mathcal{A}$ and $t \geq 0$.
3. The non-wondering set of $\mathcal{T}(\cdot)$ is given by $\Omega = \{x_1^*, \cdots, x_p^*\}$, where x_i^* is a hyperbolic equilibrium for all $i \in \{1, \ldots, p\}$.
4. $\dim W^u_{loc}(x^*) < \infty$ for every $x^* \in \Omega$, where dim is the dimension related to the differentiable manifold.
5. $W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ are transverse for all $i \neq j$, i.e., if $z \in W^u(x_i^*) \cap W^s_{loc}(x_j^*)$, then
$$T_zW^u(x_i^*) + T_zW^s_{loc}(x_j^*) = X,$$
where $T_zW^u(x_i^*)$ and $T_zW^s_{loc}(x_j^*)$ are the tangent spaces of $W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ at z, respectively.

Definition

Let (M, d) be a metric space, $\mathcal{T} : M \to M$ be a map and $\mathbb{T} = \mathbb{Z}, \mathbb{Z}^-$.

1. A sequence $\{x_n\}_{n \in \mathbb{T}}$ in M is said to be a δ**-pseudo orbit** of $\mathcal{T}$ for some $\delta > 0$ if
$$d(\mathcal{T}x_n, x_{n+1}) \leq \delta, \ \forall \ n \in \mathbb{T}.$$
2. Let $\{x_n\}_{n \in \mathbb{T}}$ and $\{z_n\}_{n \in \mathbb{T}}$ be sequences on M. We say that $\{z_n\}_{n \in \mathbb{T}}$ ϵ**-shadows** $\{x_n\}_{n \in \mathbb{T}}$, for some $\epsilon > 0$ if
$$d(x_n, z_n) \leq \epsilon, \ \forall \ n \in \mathbb{T}.$$
3. We say that $\mathcal{T}$ admits the **Shadowing property** if for any $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $\{x_n\}_{n \in \mathbb{T}}$ of $\mathcal{T}$ is ϵ-shadowed by an orbit or $\mathcal{T}$, that is, there exists $x \in M$ such that
$$d(x_n, \mathcal{T}^n x) \leq \epsilon, \forall \ n \in \mathbb{T}.$$
4. We say that $\mathcal{T}$ admits the α**-Holder Shadowing** property if there exists $d_0, L > 0$ and $\alpha \in (0, 1]$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is Ld^α-shadowed by an orbit of $\mathcal{T}$. If $\alpha = 1$ we say that $\mathcal{T}$ have the **Lipschitz Shadowing** property.
5. We say that $\mathcal{T}$ admits the **Logarithm Shadowing** property if there exists $d_0, L > 0$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is $Ld|\ln d|$-shadowed by an orbit of $\mathcal{T}$.

Main Result

Theorem

Let $\mathcal{T}(\cdot)$ be a Morse Smale semigroup on a Hilbert space X with global attractor $\mathcal{A}$ and $\mathcal{U}$ a bounded neighborhood of $\mathcal{A}$ such that $\mathcal{T}(1)|_{\mathcal{U}}$ has Lipschitz constant L_1. Then $\mathcal{T}(1)|_{\mathcal{A}}$ satisfies the Lipschitz shadowing property on $\mathcal{A}$ and:

1. *If $L_1 < 1$ then $\mathcal{T}(1)$ has the Lipschitz Shadowing property on $\mathcal{U}$;*
2. *If $L_1 = 1$ then $\mathcal{T}(1)$ has the Logarithm Shadowing property on $\mathcal{U}$;*
3. *If $L_1 > 1$ then $\mathcal{T}(1)$ has the α-Holder Shadowing property on $\mathcal{U}$, where $\alpha = \frac{\gamma}{\gamma + L_1}$ and γ is the coefficient of exponential attraction.*

Idea of the Proof

The following theorem is one the main results on this work. Here we will construct a geometric structure that will allow us later, with tools from functional analysis, to obtain the Lipschitz Shadowing on the attractor. The original idea of this Theorem can be found on the prestigious paper [1], where the author works with a C^1-Morse-Smale diffeomorphism on a compact smooth manifold (without border). The reader can also consulte the proof on [2]. This proofs are based on a induction which strongly uses the surjectiveness of the derivative, that do we not have here. To overcome this problem, we proved the auxilar lemmas 8 and **??** about the dynamics of the attractor $\mathcal{A}$. Moreover, the proof on the finite dimensional case uses (to contruct the subbundles S_i) that we can easily obtain a continuous subbundle on $\gamma^-(B)$ if we already have a continuous subbundle on B, which is not trivial in our case, when the derivative its not an isomorphism. Finally, if you have a diffeomorphism on a compact (finite dimensional) manifold you can assume without lost of generality that the constant of hiperbolicity C is equal to 1 (Lyapunov metric) and we do not have this tool [2].

Theorem

Let $\{\mathcal{T}(t) : t \geq 0\}$ be a Morse-Smale gradient semigroup and $\Omega = \{x_1^*, \cdots, x_p^*\}$ be its non-wandering set. Then, for each $i \in \{1, \ldots, p\}$ there exists $\tilde{C} > 0$, $\lambda_1 \in (0, 1)$, neighborhoods $\mathcal{O}_i$ of x_i^* in $\mathcal{A}$, subbundles S_i, U_i defined on $\overline{\mathcal{O}_i} \cup \gamma(\mathcal{O}_i)$ and continuous on $\overline{\mathcal{O}_i}$ such that

1. *For all $x \in \gamma(\mathcal{O}_i)$ it holds*
$$(D_x\mathcal{T}(t))S_i(x) \subset S_i(\mathcal{T}(t)x), \forall \ t \geq 0 \ \text{ and } \ (D_x\mathcal{T}(t))U_i(x) = U_i(\mathcal{T}(t)x), \forall \ t \in \mathbb{R}.$$
2. *$S_i(x_i^*) = \mathcal{S}(x_i^*)$, and $U_i(x_i^*) = \mathcal{U}(x_i^*)$ for all $i \in \{1, \ldots, p\}$.*
3. *$S_j(x) \subset S_k(x)$, $U_k(x) \subset U_j(x)$, for every $x \in \gamma^+(\mathcal{O}_j) \cap \gamma^-(\mathcal{O}_k)$.*
4. *$S_i(x) \oplus U_i(x) = X$, for all $x \in \gamma^-(\mathcal{O}_i)$.*
5. *For every $x \in \mathcal{O}_i$ it holds*
$$\|(D_x\mathcal{T}(t))v^s\| \leq \tilde{C}\lambda_1^t\|v^s\|, \forall \ v^s \in S_i(x), \ \forall \ 0 \leq t \leq 1, \tag{2}$$
$$\|(D_x\mathcal{T}(-t))v^u\| \leq \tilde{C}\lambda_1^t\|v^u\|, \forall \ v^u \in U_i(x), \ \forall \ 0 \leq t \leq 1. \tag{3}$$

Lemma

Given $i \in \{1, \ldots, p\}$, $k < i$, $d \in (0, 1)$ and $\epsilon, \epsilon' > 0$ there exists $\delta > 0$ such that if $x \in B(x_i^*, \delta) \cap \mathcal{A}$ satisfies

1. *there exists $t > 0$ such that $\mathcal{T}(t)x \in V^{-1}(k + d) \cap W^s(x_k^*)$*
2. $$d(\mathcal{T}(t)x, x_j^*) \geq \epsilon', \ \forall t \in \mathbb{R}, \ \forall k < j < i \tag{4}$$

then
$$d(\mathcal{T}(t)x, W^u(x_i^*)) < \epsilon.$$

Theorem

Consider the following conditions:

1. *For each $k \in \mathbb{Z}$ there exists continuous projections $P_k, Q_k : E_k \to E_k$ and constants $\lambda_1 \in (0, 1), \mathcal{C} > 0$ independent of k such that*
$$P_k + Q_k = Id, \quad \| P_k \|, \| Q_k \| \leq M, \ \forall k \in \mathbb{Z} \tag{5}$$
and
$$\| A_kP_k \|_{\mathcal{L}(E_k, E_{k+1})} \leq \lambda_1, \quad A_kP_k(E_k) \subset P_{k+1}E_{k+1}, \ \forall k \in \mathbb{Z}. \tag{6}$$
2. *There exists linear maps $B_k : Q_{k+1}E_{k+1} \to E_k$ satisfying*
$$B_kQ_{k+1}E_{k+1} \subset Q_kE_k, \ \| B_kQ_{k+1} \|_{\mathcal{L}(E_{k+1}, E_k)} \leq \lambda_1, \ A_kB_k|_{Q_{k+1}E_{k+1}} = Id_{Q_{k+1}E_{k+1}}, \ \forall k \in \mathbb{Z}. \tag{7}$$
3. *There exists $\Delta, D > 0$ with the property*
$$\| w_{k+1}v - w_{k+1}v' \| \leq D \| v - v' \|, \ \forall \| v \|, \| v' \| \leq \Delta, \ \forall k \in \mathbb{Z}. \tag{8}$$
4. *$DN_1 < 1$, where*
$$N_1 = M\left(\frac{1 + \lambda_1}{1 - \lambda_1}\right).$$
If itens (1)-(4) are fullfilled, then there exists $d_0, L > 0$ with the property: if
$$\| \phi_k(0) \| \leq d \leq d_0, \ \forall k \in \mathbb{Z} \tag{9}$$
then there exists $v_k \in E_k$ such that
$$\| v_k \| \leq Ld \ \text{ and } \ \phi_k(v_k) = v_{k+1}, \ \forall k \in \mathbb{Z}. \tag{10}$$

Theorem (Lipschitz Shadowing on $\mathcal{A}$)

Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup with non-wandering set $\Omega = \{x_1^*, \ldots, x_p^*\}$. Then $\mathcal{T}$ has the Lipschitz Shadowing property on the global attractor, that is, there exists $L, d_0 > 0$ such that for any $\{x_n\}_{n \in \mathbb{Z}}$ d-pseudo-orbit of $\mathcal{T}$ in $\mathcal{A}$, with $d \leq d_0$, there exists $x \in \mathcal{A}$ such that
$$d(\mathcal{T}^n x, x_{n+1}) \leq Ld, \ \forall k \in \mathbb{Z}.$$

After proving the Lipschitz Shadowing property on $\mathcal{A}$, we can just expand this property on a bounded neighborhood by using the attraction property of the global attractor.

Selected publications

[1] Robbin, J. W. (1971). **A structural stability theorem.** Annals of Mathematics, 99(1), 447-493.

[2] Pilyugin, S. (1999). **Shadowing in Dynamical Systems.** Springer-Verlag, volume 1706.

[3] Bortolan, M. C., Carvalho, A. N., Langa, J. A. (2020). **Attractors Under Autonomous and Non-autonomous Perturbations.** Mathematical Surveys and Monographs. AMS.

[4] Brunovsky, P., Raugel, G. (2003)**Genericity of the Morse-Smale property for damped wave equations.** Journal of Dynamics and Differential Equations.

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