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DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

ATTRACTORS FOR EVOLUTION EQUATIONS ON TIME-DEPENDENT DOMAINS

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Introduction

We present two results for dynamical systems in time-dependent phase spaces that satisfy the smoothing property. The first result consists of estimating the fractal dimension of a pullback attractors when the evolution process satisfies the smoothing property on the attractor, and the second result is about the existence of a pullback exponential attractor whenever there is a family of bounded subsets that is dissipative and positively invariant. Furthermore, we apply this theory to estimate the fractal dimension of the pullback attractor of the heat transfer equation with delay, and also to show that the 2D-Navier-Stokes equations on a non-cylindrical domain has a pullback exponential attractor.

Fractal dimension

Definition 1: Let X be a metric space and K a totally bounded subset of X. The fractal dimension of K is defined by

$$dim_f(K,X) = \limsup_{r\to 0^+} \frac{\log N_r[K,X]}{-\log r},$$

where $N_r[K,X]$ is the minimum number of balls of radius r that cover K

Theorem 1: Let $\{(X_t, \|\cdot\|_{X_t})\}_{t\in\mathbb{T}}$ and $\{(W_t, \|\cdot\|_{W_t})\}_{t\in\mathbb{T}}$ be two families of normed vector spaces and $\{\mathscr{C}_t\}_{t\in\mathbb{T}}$ be a family of bounded subsets of $\{X_t\}_{t\in\mathbb{T}}$. Assume that there is $t_0 \in \mathbb{T}$ such that W_t is compactly embedded in X_t for every $t \le t_0$, and there exists a positive constant $\varrho_0 = \varrho_0(t_0) > 0$ satisfying

$$\mathscr{C}_t \subset B_{X_t}(u_t, \varrho_0) \quad \text{for all } t \le t_0,$$

for some $u_t \in \mathscr{C}_t$. We will also suppose that there exists a family of operators $\{L_t : X_{t-1} \to W_t\}_{t\le t_0}$ with the properties

- **Negative invariance:** $\mathscr{C}_t \subset L_t \mathscr{C}_{t-1}$ for all $t \le t_0$;
- **Smoothing property:** there exists a function $\kappa : (-\infty, t_0] \to (0,+\infty)$ such that

$$\|L_t x - L_t y\|_{W_t} \le \kappa(t)\|x-y\|_{X_{t-1}}, \quad \text{for all } \; x,y \in \mathscr{C}_{t-1}, \; t \le t_0;$$

- **Entropy control:** there is an $\mathcal{N}_{t_0} \in \mathbb{N}$ such that $\sup_{s\le t_0} N^s_{1/4\kappa(s)} \le \mathcal{N}_{t_0}$, where $N^t_\epsilon := N_\epsilon[B_{W_t}(0,1), X_t]$

for any $\epsilon > 0$ and $t \le t_0$. Then,

$$\sup_{s\le t_0} dim_f(\mathscr{C}_s, X_s) \le \frac{\log \mathcal{N}_{t_0}}{\log 2}.$$

Application 1: Heat equation with memory term

Consider $\alpha, \beta \in C^2(\mathbb{R})$ with $0 < \gamma_0 \le \beta(t) - \alpha(t) \le \gamma_1$ for all $t \in \mathbb{R}$, and denote by $\mathcal{I}_t := \big(\alpha(t), \beta(t)\big)$. Then, given $\tau \in \mathbb{R}$, the heat transfer equation with delay on the non-cylindrical domain and with homogeneous Cauchy-Dirichlet boundary conditions, denoted by (DHE), is:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + g(u) = f(t) + u\big(\widetilde{r}_t, t-\delta(t)\big) & \text{in } \cup_{t\in(\tau,+\infty)} \mathcal{I}_t \times \{t\}, \\ u(\alpha(t),t) = u(\beta(t),t) = 0 \quad \forall t \in [\tau, +\infty), & \\ u(\tau) = u^\tau \quad \text{in } \mathcal{I}_\tau, & \\ u(\tau+s) = \phi(s) \quad \text{in } \mathcal{I}_{\tau+s} \text{ and } s \in (-h,0), & \end{cases} \tag{DHE}$$

where u is the temperature function, u^τ is the initial condition, ϕ is the initial condition with memory defined on interval $(-h,0)$ with $h>0$, and $f \in L^1_{loc}(Q_\tau)$ is the heat source, $r_t(x)\frac{x-\alpha(t)}{\gamma(t)}$ and $\widetilde{r}_t := r^{-1}_{t-\delta(t)} \circ r_t$, $\delta(t) \in C^1(\mathbb{R})$ with $\delta(t) \ge 0$, $h = \sup_{t\in\mathbb{R}} \delta(t) > 0$ and $\delta_* = \sup_{t\in\mathbb{R}} \delta'(t) < 1$, and $g \in C^1(\mathbb{R})$ is such that there exist $\alpha_0, \alpha_1, \beta_0 \ge 0$ and l, and $p \ge 2$ such that

$$-\beta_0 + \alpha_0 |s|^p \le g(s)s \le \beta_0 + \alpha_1|s|^p \quad \text{and} \quad g'(s) \ge -l \qquad \forall s \in \mathbb{R}.$$

Definition 2: (Tempered universes on $M^2_{L^2}(\mathcal{I}_t)$) Given a non-increasing function $\eta : \mathbb{R} \to (0,+\infty)$, we define by $\mathcal{D}_\eta(M^2_{L^2})$ the class of all families $\widehat{D} = \{D(t) \subset M^2_{L^2}(\mathcal{I}_t) : t \in \mathbb{R},\ D(t) \neq \emptyset\}$ such that

$$\lim_{\tau\to-\infty} e^{\eta_t \tau} \sup_{(v,\phi)\in D(\tau)} \|(v,\phi)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)} = 0, \ \forall t \in \mathbb{R}.$$

For a family $\{X_t\}$ we are going to consider $f \in L^2_{loc}(\mathbb{R}; X_t)$ satisfying

$$M_f(t) := \sup_{s\le t} \int_{s-1}^{s} \|f(r)\|^2_{X_t} dr < +\infty, \quad \text{for all } t \in \mathbb{R}. \tag{1}$$

Theorem 2: *(Fractal Dimension)* Suppose that conditions (H1)-(H4) hold, and $g \in C^1(\mathbb{R})$ and $p > 2$. Assume that there exists a non-increasing bounded function $\eta : \mathbb{R} \to (0,+\infty)$ such that $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ satisfies (1). Then, the pullback attractor $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$ has finite fractal dimension.

proof: Given $t \in \mathbb{R}$ consider the Banach space $W_{L^2;H^1_0}(\mathcal{I}_t) = H^1_0(\mathcal{I}_t) \times \Big\{\phi \in L^2_{H^1_0}(\mathcal{I}_t) : \phi' \in L^2_{L^2}(\mathcal{I}_t)\Big\}$. Then, it follows from Lemma of *Aubin-Lions-Simon* that

$$W_{L^2;H^1_0}(\mathcal{I}_t) \hookrightarrow\hookrightarrow M^2_{L^2}(\mathcal{I}_t) \quad \text{for all } t \in \mathbb{R}.$$

There is a sequence of points $\{\hat{t}_{-m}\}_{m\le 0}$ such that $\hat{t}_{-m} \to -\infty$, and $S_m := S(\hat{t}_{-m}, \hat{t}_{-m+1})$ satisfies

$$\|S_m(u,\phi) - S_m(v,\varphi)\|_{W_{L^2;H^1_0}(\mathcal{I}_{\hat{t}_{-m}})} \le \tilde{\kappa}_h \|(u,\phi)-(v,\varphi)\|_{M^2_{L^2}(\mathcal{I}_{\hat{t}_{-m+1}})},$$

for all $(u,\phi) \in \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(\hat{t}_{-m+1})$ and $m \le 0$. Since the pullback attractor is invariant and the evolution process $S(\cdot,\cdot)$ is Lipschitz, it follows from Theorem 1 that

$$\sup_{t\in\mathbb{R}} dim_f\left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t), M^2_{L^2}(\mathcal{I}_t)\right) \le \frac{\log\left[N_{M^2_{L^2}(\mathbf{I}_{\hat{t}_0})}\Big(B_{W_{L^2;H^1_0}(\mathbf{I}_{\hat{t}_0})}(0,1), \frac{1}{4\tilde{\kappa}_h}\Big)\right]}{\log 2}.$$

Pullback exponential attractor

Theorem 3: *(Pullback exponential attractor)* Let $t_0 \in \mathbb{R}$ and $\mathcal{S}(\cdot,\cdot)$ be an t_0-evolution process defined on the family of Banach spaces $\{X_t\}_{t\le t_0}$, and let $\{M_t\}_{t\le t_0}$ be a family of bounded closed subsets of $\{X_t\}_{t\le t_0}$. Assume that:

($\mathcal{C}1$) There exists $T > 0$ such that $\mathcal{S}(t, t-T)M_{t-T} \subset M_t$ for all $t \le t_0$;

($\mathcal{C}2$) For each $t \le t_0$ there exists $c_t > 0$ such that $M_s \subset B_{X_s}(0, c_t)$, for all $s \le t \le t_0$;

($\mathcal{C}3$) There exists a family $\{Z_t\}_{t\le t_0}$ of Banach spaces such that $Z_t \hookrightarrow\hookrightarrow X_t$ for each $t \le t_0$;

($\mathcal{C}4$) For $s \le t \le t_0$ there exists $\kappa(t, t-s) > 0$ such that

$$\|\mathcal{S}(t,s)x_1 - \mathcal{S}(t,s)x_2\|_{Z_t} \le \kappa(t,t-s)\|x_1 - x_2\|_{X_s}, \quad \text{for all } x_1, x_2 \in M_s;$$

($\mathcal{C}5$) There exists $\mathcal{N}_{t_0} > 0$ such that $\sup_{s\le t_0} N(s) \le \mathcal{N}_{t_0}$, where

$$N(s) := N_{X_s}\left(B_{Z_s}(0,1), \frac{1}{4\kappa_s}\right) \quad \text{with } \kappa_s := \kappa(s,1) \quad \text{for all } s \le t_0;$$

($\mathcal{C}6$) For $t \le t_0$ and $s \in [0,T]$ there exists $L_{t,t-s} > 0$ such that

$$\|\mathcal{S}(t,t-s)x_1 - \mathcal{S}(t,t-s)x_2\|_{X_t} \le L_{t,t-s}\|x_1 - x_2\|_{X_{t-s}}, \quad \text{for all } x_1, x_2 \in M_{t-s}.$$

Then there exists a family $\{\mathcal{M}_t\}_{t\le t_0}$ of compact subsets of X_t with the properties:

- *(Positively Invariance):* $\mathcal{S}(t,s)\mathcal{M}_s \subset \mathcal{M}_t$, for all $s \le t \le t_0$.
- *(Pullback Exponential Attractability):* There exists $\beta > 0$ such that

$$\lim_{s\to+\infty} e^{s\beta}\, \text{dist}_{X_t}\big(\mathcal{S}(t,t-s)M_{t-s}, \mathcal{M}(t)\big) = 0, \quad \text{for all } t \le t_0.$$

- *(Finite Fractal Dimension):* It holds

$$\sup_{t\le t_0} d_f^{X_t}(\mathcal{M}_t) \le \frac{\log \mathcal{N}_{t_0}}{\log(2)} < \infty.$$

Application 2: 2D Navier-Stokes equations

Let us consider the family $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ of open bounded subset of $\mathbb{R}^2$. Now, we are going to assume some hypotheses of regularity on the family of open sets $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ within the framework of a correct formulation of our problem. Let us denote by $\mathcal{O}$ a nonempty bounded open subset of $\mathbb{R}^2$ and consider the map $r(\cdot,\cdot) = r(y,t)$ a vector function $r \in C^1(\overline{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^2)$ such that $r(\cdot,t) : \mathcal{O} \to \mathcal{O}_t$ is C^3- diffeomorphism . Denote by $\overline{r}(\cdot,t) := r^{-1}(\cdot,t)$. Suppose **(H1)** $Jac(\overline{r},x,t) = det\Big(\frac{\partial \overline{r}_i}{\partial x_j}(x,t)\Big) \equiv \frac{1}{J(t)} > 0$, **(H2)** $\overline{r} \in C^{3,1}\Big(\overline{\bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\}}; \mathbb{R}^2\Big)$ and **(H3)** $\Omega_t := \bigcup_{s\le t} \mathcal{O}_t$ is bounded in $\mathbb{R}^2$ for all $t \in \mathbb{R}$.

Then, we call the 2D Navier-Stokes system, indicated by **(NS)**, to the system given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u\cdot\nabla)u + \nabla\pi = f(t) & \text{in} \quad \cup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}, \\ \text{div } u = 0 & \text{in} \quad \cup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}, \\ u \equiv 0 & \text{on} \quad \cup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}, \\ u(\tau) = u_\tau & \text{in} \quad \mathcal{O}_\tau, \end{cases} \tag{NS}$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ is the velocity field, u_τ is the given initial velocity vector field, the function $\pi(x,t)$ represents the pressure on the fluid, and $f(x,t) = (f_1(x,t), f_2(x,t))$ is a external force. Let us denote by

$$\begin{cases} \mathcal{V}_t := \{\varphi \in C_c^\infty(\mathcal{O}_t)^2 : \text{div}\,\varphi = 0\}; \quad H_t = \text{ the closure of } \mathcal{V}_t \text{ in the } L^2(\mathcal{O}_t)^2 - \text{ norm;} \\ V_t = \text{ the closure of } \mathcal{V}_t \text{ in the } W^{1,2}(\mathcal{O}_t)^2 - \text{ norm.} \end{cases}$$

Theorem 4: *(Existence and Uniqueness)* Suppose that hypotheses (H1)-(H2) hold. Let us consider τ, T with $\tau < T$. Then, for any $u^\tau \in H_\tau$, and $f \in L^2(\tau, T; V_t^*)$, there exists a unique weak solution of the problem **(NS)**.

Definition 3: (Tempered universe on H_t) Let us denote by $\mathcal{D}^H_\lambda$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) \subset H_t : t \in \mathbb{R},$ and $D(t) \neq \emptyset\}$ such that, for all $t \in \mathbb{R}$

$$\lim_{\tau\to-\infty} e^{\lambda_{1,\tau}\tau} \sup_{v\in D(\tau)} |v|^2_\tau = 0.$$

Theorem 5: Under assumptions (H1)-(H3) and $f \in L^2_{loc}(\mathbf{R}; H_t)$ satisfying (1), there exists a family $\mathcal{M} = \{\mathcal{M}(t) \subset X_t : t \in \mathbb{T}\}$, that is a pullback $\mathcal{D}^H_\lambda$-exponential attractor for the process $\mathcal{S}(\cdot,\cdot)$.

proof: Let us consider the family of bounded and closed sets $\widehat{M} = \{M(t) : t \in \mathbb{R}\}$ defined as

$$M(t) = \overline{\bigcup_{s\in[\tau(t,\widehat{B}_0),t]} S(t,s)B_{V_s}[0, C_f(s)]}^{V_t},$$

Then, the evolution process satisfies

$$\|\mathcal{S}(t,s)u - \mathcal{S}(t,s)v\|_t \le \kappa_{t,t-s}|u-v|_s, \quad \text{for all } u,v \in M(s), \ s \le t.$$

Then, the dynamical system $(\mathcal{S}(\cdot,\cdot), \{H_t\}_{t\in\mathbb{R}})$ satisfies the hypotheses ($\mathcal{C}1$)-($\mathcal{C}6$). Therefore, the existence of a pullback $\mathcal{D}^H_\lambda$-exponential attractor is given by Theorem 3.

Selected publications

[1] A. N. Carvalho, H. López-Lázaro, J. Huaccha-Neyra **Smoothing property of an evolution process associated with semilinear heat equation with delay on an interval with moving ends** (2024) submitted.

[2] H. López-Lázaro, M. J. D. Nascimento, O. Rubio. **Finite fractal dimension of pullback attractors for semilinear heat equation with delay in some domain with moving boundary** *Nor.* *Ar*

[3] H. López-Lázaro, M. J. D. Nascimento, C. Takaessu Jr., V. T.Azevedo **Pullback attractors with finite fractal dimension for a semilinear** **e** 393 (2024)

[4] J. Huaccha-Neyra, H. López-Lázaro, O. Rubio, C. Takaessu Jr. **Pullback exponential attractor of dynamical systems associated with non-cylindrical problems** (2023) submitted

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