Some results on the asymptotic dynamics of non-dissipative problems: unbounded attractors

M. Belluzi[1,2], M. Bortolan, U. Castro and J. Fernandes

(1) Instituto de Ciências Matemáticas e de Computação, USP.
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Introduction

The study of unbounded attractors started with the papers of [4, 5] in the context of abstract evolution equations, specifically of parabolic semilinear equations as the reaction-diffusion equation with non-dissipative non-linearities. It was only recently that this object was proposed in the context of semigroup, in [3]. In the sequel, we discuss the permanence of this unbounded attractor subjected to perturbations.

Unbounded attractor for nonlinear semigroups

Definition *A closed set $\mathcal{U} \subset X$ is called an* ***unbounded attractor*** *for T if*

1. *$T(t)\mathcal{U} = \mathcal{U}$ for all $t \geqslant 0$ (invariance);*
2. *For all $B \subset X$ bounded we have $\lim_{t\to\infty} d_H(T(t)B, \mathcal{U}) = 0$ (attraction);*
3. *There is no proper closed subset of $\mathcal{U}$ satisfying both 1 and 2.*

Characterization of the unbounded attractor: In certain situations,the unbounded attractor is unique and given as the set of *bounded in the past global solutions*

$$\mathcal{J} = \{\xi(0) \colon \xi \text{ is a bounded in the past global solution of } T\}.$$

Existence of the unbounded attractor: To ensure the existence of this object, we need the following conditions on T.

Definition *A semigroup T in a Banach space X is* ***u-asymptotically compact*** *if for each $B \subset X$ bounded there exists $t_0 = t_0(B) \geqslant 0$ such that there exists a family of compact sets $\{K(t) \subset X\}_{t\geq t_0}$ with*

$$\lim_{t\to\infty} d_H(T(t)B, K(t)) = 0.$$

Definition *Let T be a semigroup. We say that a set $G \subset X$ is* ***u-strongly absorbing*** *for T if*

1. *$T(t)G \subset G$ for all $t \geqslant 0$ (positive invariance);*
2. *for each $B \subset X$ bounded there exists $t_0 = t_0(B) \geqslant 0$ such that $T(t)B \subset G$ for all $t \geqslant t_0$;*
3. *there exists a sequence of bounded sets $\{H_n\}_{n\in\mathbb{N}} \subset G$ such that:*
   - *$H_n \subset H_{n+1}$ for each $n \in \mathbb{N}$;*
   - *$G \setminus H_n$ is positively invariant by T for each $n \in \mathbb{N}$;*
   - *if $B \subset G$ is bounded there exists $n \in \mathbb{N}$ such that $B \subset H_n$.*
4. $\lim_{t\to\infty} d_H(T(t)G, \mathcal{J}) = 0.$

Theorem *[3] If T is u-asymptotically compact and has an u-strongly absorbing set G, than $\mathcal{J}$ is the unique unbounded attractor for T. Moreover, $\mathcal{J} \subset G$ and $\mathcal{J}$ is bounded-compact, that is, $\mathcal{J} \cap F$ is compact for each closed and bounded subset F of X.*

Proposition *[3] Let T be a semigroup such that the set $\mathcal{J}$ is the unique unbounded attractor, and assume that the collection of equilibria of T is finite, $\mathcal{E} = \{u_1, \ldots, u_n\}$. If T is gradient with respect to $\mathcal{E}$, then $\mathcal{J} = \bigcup_{i=1}^n W^u(u_i)$.*

Fractional reaction-diffusion equations

Consider the fractional versions of the scalar-reaction diffusion equation

$$\partial_t u + (-\partial_{xx})^\alpha u = bu + g(x,u), \quad x \in (0,\ell), \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where $\alpha \in (1-\delta, 1+\delta)$, with $b > 0$ and $\delta > 0$ a parameter sufficiently small.

Abstract formulation: Let $A : H^2(0,\ell) \cap H^1_0(0,\ell) \subset L^2(0,\ell) \to L^2(0,\ell)$ be the operator $A = -\partial_{xx}$ and $\tilde{g}(u)(x) = g(u(x))$ the Nemitskiĭ operator associated to g. We assume that g is bounded and globally Lipschtiz, uniformly in $x \in (0,\ell)$. We have A *positive* and *self-adjoint* with *compact resolvent*

$$\sigma(A) = \{\lambda_j \colon \quad 0 < \lambda_1 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots\}_{j\in\mathbb{N}} \quad \text{and} \quad A\varphi_j = \lambda_j\varphi_j, \quad \varphi \in D(A).$$

Problem (1) can be rewritten as an abstract evolution equation in $X = L^2(\Omega)$

$$u_t = (-A^\alpha + bI)u + \tilde{g}(u) = L_\alpha u + \tilde{g}(u), \quad t > 0, \tag{2}$$

where $L_\alpha := -A^\alpha + bI$.

Lemma *$\sigma(L_\alpha) = \{-\lambda_j^\alpha + b\}$ and φ_j is the eigenvalue of L_α associated to $-\lambda_j^\alpha + b$. Moreover, L_α is the infinitesimal generator of an analytic semigroup $\{e^{L_\alpha t} \colon t \geqslant 0\}$ in $X = L^2(\Omega)$ and problem (2) is globally well posed in X. We denote by $T_\alpha(t)$ the semigroup obtained by (2).*

Hypothesis: There exists $N \in \mathbb{N}$, $\sigma > 0$ and $\delta > 0$ sufficiently small, such that for all $\alpha \in (1-\delta, 1+\delta)$, we have

$$\lambda_N^\alpha < b < \lambda_{N+1}^\alpha \quad \text{and} \quad 0 < \sigma < \min\{b - \lambda_N^\alpha, \lambda_{N+1}^\alpha\}. \tag{3}$$

Splitting the space: $X = L^2(0,\ell)$, $X = E_N \oplus F_N$, and any $u \in X$ is given by $u = p + q$

$$E_N = [\varphi_1, \ldots, \varphi_N] \quad \text{and} \quad F_N = E_N^\perp$$

$$e^{L_\alpha t}|_{E_N} \text{ grows exponentially} \quad \text{and} \quad e^{L_\alpha t}|_{F_N} \text{ decays exponentially} \tag{4}$$

Absorbing set and u-asymptotically compactness: under the additional assumption

$$\|\tilde{g}(p+q)\| \overset{\|p\|\to\infty}{\longrightarrow} 0, \qquad \text{uniformly for } q \text{ in bounded sets of } F_N$$

Proposition *There exists a constant $D > 0$, independent of $\alpha \in (1-\delta, 1+\delta)$, such that*

$$G := \{u = p + q \in E_N \oplus F_N \colon \|q\| \leqslant D\}$$

is u-strongly absorbing for T_α for all $\alpha \in (1-\delta, 1+\delta)$.

Theorem *The set $\mathcal{J}_\alpha = \{\xi_\alpha(0); \xi_\alpha : \mathbb{R} \to X$ is bounded global solution of $T_\alpha\}$ is the only unbounded attractor of T_α, $\mathcal{J}_\alpha \subset G$ and $\mathcal{J}_\alpha$ is bounded compact.*

Continuity of the Unbounded Attractors

Fix $R > 0$. For $\alpha \in (1-\delta, 1+\delta)$, the set $J_{\alpha,R}$ below is compact (from the bounded-compactness of $\mathcal{J}_\alpha$):

$$J_{\alpha,R} = \mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| \leqslant R\}. \tag{5}$$

Proposition *[1] Given $\{x_n\}_{n\in\mathbb{N}} \subset X$ bounded with $x_n \in \mathcal{J}_{\alpha_n}$ for each $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence to a point $x \in \mathcal{J}_\alpha$. In particular, for each $R > 0$ and $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\lim_{\beta\to\alpha} d_H(J_{\beta,R}, J_{\alpha,R}) = 0, \tag{6}$$

where $J_{\beta,R}$ and $J_{\alpha,R}$ are as in (5)*.*

Lemma *[1] Given $\varepsilon > 0$ there exists $R > 0$ such that for all $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| > R\} \subset \mathcal{O}_\varepsilon(E_N).$$

Moreover, for each $p \in E_N$ and $\alpha \in (1-\delta, 1+\delta)$ there exists $q = q(\alpha) \in F_N$ such that $p + q \in \mathcal{J}_\alpha$.

Theorem (Upper semicontinuity) *[1] For each $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\lim_{\beta\to\alpha} d_H(\mathcal{J}_\beta, \mathcal{J}_\alpha) = 0.$$

Lower Semicontinuity: problem (2) is gradient with Lyapunov function given by $V_\alpha \colon \mathcal{J}_\alpha \to \mathbb{R}$

$$V_\alpha(u) = \|A^{\frac{\alpha}{2}}u\|^2 - b\|u\|^2 - 2\int_0^\ell \int_0^{u(x)} g(x,s)\,ds\,dx. \tag{7}$$

Proposition *[1] Assume that there exists $\alpha \in (1-\delta, 1+\delta)$ such that $\mathcal{E}_\alpha$ consists exactly of n hyperbolic equilibria $e_{\alpha,1}, \ldots, e_{\alpha,n}$. Then there exists $\mu_\alpha > 0$ such that for $\beta \in (\alpha - \mu_\alpha, \alpha + \mu_\alpha) \subset (1-\delta, 1+\delta)$, $\mathcal{E}_\beta$ consists exactly of n hyperbolic equilibria $e_{\beta,1}, \ldots, e_{\beta,n}$ with*

$$\max_{i=1,\ldots,n} \|e_{\beta,i} - e_{\alpha,i}\| \to 0 \quad \text{as } \beta \to \alpha.$$

For $\rho > 0$ and $e_{\alpha,i} \in \mathcal{E}_\alpha$, we define the **$\rho$-local unstable manifold** of $e_{\alpha,i}$ as the set

$$W^u_\rho(e_{\alpha,i}) = \{\xi(0) \colon \xi \text{ is a global solution of } T_\alpha \text{ with } \|\xi(t) - e_{\alpha,i}\| < \rho \text{ for all } t \leqslant 0 \text{ and } \xi(t) \to e_{\alpha,i} \text{ as } t \to -\infty\}.$$

Proposition *[2] For ρ sufficiently small and $\alpha \in (1-\delta, 1+\delta)$, the local unstable manifolds behave continuously, that is, $d_H(W^u_\rho(e_{\beta,i}), W^u_\rho(e_{\alpha,i})) \to 0$, as $\beta \to \alpha$.*

Theorem (Lower semicontinuity) *[1] For each $\alpha \in (1-\delta, 1+\delta)$, we have*

$$\lim_{\beta\to\alpha} d_H(\mathcal{J}_\alpha, \mathcal{J}_\beta) = 0.$$

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