Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Friedrich-Alexander-Universität
DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

DISSIPATIVITY AND STABILITY OF HIGHER-INDEX ABSTRACT DIFFERENTIAL-ALGEBRAIC EQUATIONS AND APPLICATIONS

Maria Filipkovska

Chair for Dynamics, Control, Machine Learning and Numerics – AvH Professorship, FAU Erlangen-Nürnberg

Introduction

In this paper, we deals with partial differential equations (PDEs) and the systems of PDEs and algebraic equations, which are commonly referred to as partial differential-algebraic equations (PDAEs), that can be represented as an implicit differential equation of the form

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \qquad t > 0, \tag{1}$$

where A and B are closed linear operators from X into Y with domains D_A and D_B respectively, $D = D_A \cap D_B \neq \{0\}$ is a lineal (linear manifold), X and Y are Banach spaces, D_A and D_B are dense in X (i.e., $\overline{D_A} = X$, $\overline{D_B} = X$), and $f \in C(\mathbb{R}_+ \times D, Y)$, $\mathbb{R}_+ = [0, \infty)$. The operator A is degenerate and the operator B can also be degenerate. Equation (1) with the degenerate operator A is called a degenerate differential equation (degenerate DEs), an abstract evolution equation or an *abstract differential-algebraic equation* (abstract DAE). A significant feature of these equations is that any type of a PDE can be represented as an abstract evolution equation in appropriate infinite-dimensional spaces, possibly, with a complementary boundary condition, and it is often much easier to study the corresponding abstract evolution equation than the original PDE.

We present conditions of dissipativity, stability and asymptotic stability for higher-index abstract DAEs. It is known that, from a practical point of view, stability is the most important property of an engineering system since it guarantees that the system can work properly and will not be destroyed. In control theory, the dissipativity and complete stability (asymptotic stability in the large of an equilibrium state) of a dynamical system (control system with zero input) are used to obtain conditions for the stability of the associated control system and to analyze the robustness of the control system with respect to disturbance. DAEs arise from the modelling of various systems and processes in control problems, gas industry, mechanics, radio engineering, chemical kinetics, economics, ecology and biology.

Direct decompositions of spaces and the associated projectors

We assume that the operator pencil $P(\lambda) = \lambda A + B \colon D \to Y$ ($\lambda \in \mathbb{C}$ is a parameter) is regular, i.e., the set of its regular points $\varrho = \varrho(A, B) = \{\lambda \in \mathbb{C} \mid \exists\, (\lambda A + B)^{-1} \in \mathrm{L}(Y, \overline{D})\}$ is not empty.

Definition. Let the following conditions hold:
(a) The pencil $P(\lambda) = \lambda A + B$ is regular for all λ from some neighborhood of the infinity.
(b) The point $\lambda = \infty$ is a pole of the resolvent $R(\lambda) = P^{-1}(\lambda) = (\lambda A + B)^{-1}$ of order r; this is equivalent to the fact that the resolvent $\widehat{R}(\mu) = (A + \mu B)^{-1}$ has a pole of order $\nu = r + 1$ at $\mu = 0$.
Then $P(\lambda)$ is called a *regular pencil of index* ν ($\nu \in \mathbb{N}$).
If there exists $A^{-1} \in \mathrm{L}(Y, X)$ (or $\mu = 0$ is a regular point of $\widehat{P}(\mu)$) and $D_B \supseteq D_A$, then $P(\lambda)$ is a regular pencil of *index* 0.

Let $P(\lambda)$ be a regular pencil of index ν. Then there exists the pair of mutually complementary projectors $P_k \colon D \to D_k$ ($P_1 D_A = P_1 D$), $k = 1, 2$, and the pair of mutually complementary projectors $Q_k \colon Y \to Y_k$, $k = 1, 2$, which generate the decompositions of D and Y into the direct sums

$$D = D_1 \dot{+} D_2, \quad Y = Y_1 \dot{+} Y_2, \quad D_k = P_k D, \quad Y_k = Q_k Y, \quad k = 1, 2, \tag{2}$$

such that the pairs $\{D_k, Y_k\}$ are invariant under the operators A, B, i.e., $AD_k \subset Y_k$ and $BD_k \subset Y_k$, $k = 1, 2$ (see, e.g., [8]). The restricted operators $A_k := A\big|_{D_k} \colon D_k \to Y_k$, $B_k := B\big|_{D_k} \colon D_k \to Y_k$, $k = 1, 2$, are such that there exist $A_1^{-1} \in L(Y_1, \overline{D_1})$ and $B_2^{-1} \in L(Y_2, \overline{D_2})$. The projectors can be constructively determined by using contour integration [8] or residues [6].

If $P(\lambda)$ has index 1, the projectors mentioned above allow one to reduce equation (1) to a system of an explicit ODE and an algebraic equation. However, for the pencil of index higher than 1, additional decompositions of the lineal D_2, the subspace Y_2 and the projectors P_2, Q_2 are required. Below we provide another method to construct the projectors.

In what follows, we suppose that $D_B \supseteq D_A$, then $D = D_A$. Assume that $\dim\ker A = n$ and denote by $\{\varphi_1^1, \ldots, \varphi_n^1\}$ a basis of $\ker A$.

There exists a *canonical system* $\{\varphi_i^j\}_{i=1,\ldots,n}^{j=1,\ldots,m_i}$ ($1 \le m_i \le \nu$) *of eigenvectors and adjoined vectors of the pencil* $\widehat{P}(\mu) = A + \mu B$ *that correspond to the eigenvalue* $\mu = 0$, for which the vectors satisfy the equalities

$$A\varphi_i^1 = 0, \quad A\varphi_i^j = -B\varphi_i^{j-1}, \quad i = 1, \ldots, n, \quad j = 2, \ldots, m_i, \quad m_i \le \nu \; (\nu = r + 1), \quad \max_{i=1,\ldots,n}\{m_i\} = \nu,$$

where ν is the index of $P(\lambda)$ [7]. The vectors $\{\varphi_i^j\}_{i=1,\ldots,n}^{j=1,\ldots,m_i}$ and $\{B\varphi_i^j\}_{i=1,\ldots,n}^{j=1,\ldots,m_i}$ form the bases of D_2 and Y_2, respectively. The number $m_i - 1$ is called the *order* of the adjoined vector $\varphi_i^{m_i}$, and m_i is called the *multiplicity* of the eigenvector φ_i^1. If the eigenvector φ_i^1 does not have adjoined vectors (in this case, $\langle B\varphi_i^1, q_i^1\rangle = 1$, $q_i^1 \in \ker A^*$) then its multiplicity $m_i = 1$. The projectors P_k, Q_k can be obtained as

$$P_2 x = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \langle x, B^* q_i^j\rangle \varphi_i^j, \quad Q_2 y = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \langle y, q_i^j\rangle B\varphi_i^j, \quad P_1 = I_X - P_2, \quad Q_1 = I_Y - Q_2, \tag{3}$$

where $x \in X$, $y \in Y$ and the bounded linear functionals $q_i^j \in Y^*$ are chosen such that

$$A^* q_i^{m_i} = 0, \quad A^* q_i^j = -B^* q_i^{j+1}, \quad i = 1, \ldots, n, \; j = 1, \ldots, m_i - 1,$$

and $\langle B\varphi_i^j, q_k^l\rangle = \delta_{ik}\delta_{jl}$. The projectors (3) generate the direct decompositions $X = X_1 \dot{+} X_2$, $X_k = P_k X$, $D = D_1 \dot{+} D_2$, $D_k = P_k D$ ($D_2 = X_2$, $D_1 = X_1 \cap D$), $Y = Y_1 \dot{+} Y_2$, $Y_k = Q_k Y$, $k = 1, 2$, where the direct decompositions of D and Y are the same as (2).

Define $D_{20} := \ker A = \mathrm{span}\{\varphi_i^1\}_{i=1,\ldots,n}$, by D_{2s} the linear span of the adjoined vectors of order s ($s = 1, \ldots, \nu - 1$), by D_{20}^j the linear span of the eigenvectors of multiplicity j ($j = 1, \ldots, \nu$), by D_{2s}^j the linear span of the adjoined vectors of order s to which the eigenvectors of multiplicity j correspond ($j = s + 1, \ldots, \nu$), and $Y_{2s} := BD_{2s}$, $Y_{2s}^j := BD_{2s}^j$, $j = s + 1, \ldots, \nu$, $s = 0, \ldots, \nu - 1$. Then $D_2 = D_{20} \dot{+} \ldots \dot{+} D_{2(\nu-1)}$, $Y_2 = Y_{20} \dot{+} \ldots \dot{+} Y_{2(\nu-1)}$, $D_{2s} = D_{2s}^{s+1} \dot{+} \ldots \dot{+} D_{2s}^{\nu}$ and $Y_{2s} = Y_{2s}^{s+1} \dot{+} \ldots \dot{+} Y_{2s}^{\nu}$, $s = 0, \ldots, \nu - 1$. Introduce the projectors $P_{2s} \colon X \to D_{2s}$, $Q_{2s} \colon Y \to Y_{2s}$, $P_{2s}^{(j)} \colon X \to D_{2s}^j$, $Q_{2s}^{(l)} \colon Y \to Y_{2s}^l$. Denote $D_{2\Sigma} = D_2 \setminus \ker A = \mathrm{span}\{\varphi_i^j,\ j = 2, \ldots, m_i,\ i \in \{1, \ldots, n\} : m_i \neq 1\}$, $Y_{2*} = \mathrm{span}\{B\varphi_i^{m_i}\}_{i=1,\ldots,n}$, $Y_{2\Sigma} = Y_2 \setminus Y_{2*} = \mathrm{span}\{B\varphi_i^j,\ j = 1, \ldots, m_i - 1,\ i \in \{1, \ldots, n\} : m_i \neq 1\}$.
Then

$$D_2 = D_{20} \dot{+} D_{2\Sigma}, \quad Y_2 = Y_{2*} \dot{+} Y_{2\Sigma}, \tag{4}$$

$A_{20} = A\big|_{D_{20}} = 0$ and $A_{2\Sigma} := A\big|_{D_{2\Sigma}} \colon D_{2\Sigma} \to Y_{2\Sigma}$ has the inverse $A_{2\Sigma}^{-1} \in \mathrm{L}(Y_{2\Sigma}, D_{2\Sigma})$. Here it is supposed that $\nu \ge 2$. If $\nu = 1$, then $D_2 = \ker A$ and $Y_2 = BD_2$. The operators $A_1 = A\big|_{D_1}$, $B_2 = B\big|_{D_2}$ are invertible, as mentioned above. The direct decompositions (4) generate the pairs P_{20}, $P_{2\Sigma}$ and Q_{2*}, $Q_{2\Sigma}$ of the mutually complementary projectors $P_{20} \colon D \to D_{20}$, $P_{2\Sigma} \colon D \to D_{2\Sigma}$ and $Q_{2*} \colon Y \to Y_{2*}$, $Q_{2\Sigma} \colon Y \to Y_{2\Sigma}$. The above projectors can be obtained from (3) as appropriate partial sums.

Main results

Consider the case when $P(\lambda)$ is a regular pencil of *index 2*. In this case, $D_2 = D_{20} \dot{+} D_{2\Sigma}$, $D_{20} = D_{20}^1 \dot{+} D_{20}^2$, $D_{2\Sigma} = D_{21}$, and $Y_2 = Y_{2*} \dot{+} Y_{2\Sigma} = Y_{20} \dot{+} Y_{21}$, $Y_{2*} = Y_{20}^1 \dot{+} Y_{21}$, $Y_{2\Sigma} = Y_{20}^2$, $Y_{20} = Y_{20}^1 \dot{+} Y_{20}^2$. There exists $A_{2\Sigma}^{-1} = A_{21}^{-1} \in \mathrm{L}(Y_{20}^2, D_{21})$, where $A_{2\Sigma} = A_{21} = A\big|_{D_{2\Sigma} = D_{21}}$. Any element $x \in D$ can be uniquely represented in the form

$$x = x_1 + x_2 = x_1 + x_{20} + x_{21}, \quad x_{20} = x_{20}^{(1)} + x_{20}^{(2)}, \quad x_i = P_i x, \quad x_{2i} = P_{2i} x, \quad x_{20}^{(i)} = P_{20}^{(i)} x, \quad i = 1, 2.$$

Using the projectors Q_1, $Q_{20}^{(2)}$, $Q_{20}^{(1)}$, Q_{21}, we reduce the DAE (1) to the equivalent system

$$A_1 \dot{x}_1 + B_1 x_1 = Q_1 f(t, x), \tag{5}$$
$$A_{21} \dot{x}_{21} + B_2 x_{20}^{(2)} = Q_{20}^{(2)} f(t, x), \tag{6}$$
$$B_2 x_{20}^{(1)} = Q_{20}^{(1)} f(t, x), \tag{7}$$
$$B_2 x_{21} = Q_{21} f(t, x). \tag{8}$$

The derivative $\dot{V}_{(5),(6)}(t, x_1, x_{21})$ of the a scalar function $V \in C^1(\mathbb{R}_+ \times D_1 \times D_{21}, \mathbb{R})$ along the trajectories of the system (5), (6) has the form

$$\begin{aligned}\dot{V}_{(5),(6)}(t, x_1, x_{21}) &= \partial_t V(t, x_1, x_{21}) + \partial_{(x_1, x_{21})} V(t, x_1, x_{21}) \cdot \Pi(t, x) = \partial_t V(t, x_1, x_{21}) + \\ &+ \partial_{x_1} V(t, x_1, x_{21}) A_1^{-1}[Q_1 f(t, x) - B_1 x_1] + \partial_{x_{21}} V(t, x_1, x_{21}) A_{21}^{-1}[Q_{20}^{(2)} f(t, x) - B_2 x_{20}^{(2)}],\end{aligned} \tag{9}$$

$$\Pi(t, x) = \begin{pmatrix} A_1^{-1}[Q_1 f(t, x) - B_1 x_1] \\ A_{21}^{-1}[Q_{20}^{(2)} f(t, x) - B_2 x_{20}^{(2)}] \end{pmatrix}.$$

We will study the initial value problem (IVP) for the DAE (1) with the initial condition

$$x(0) = x_0. \tag{10}$$

Theorem 1. Let $f \in C(\mathbb{R}_+ \times D, Y)$, $D_B \supseteq D_A$, $\dim\ker A = n < \infty$, and $\lambda A + B$ be a regular pencil of index 2. Assume that there exists an open set $M_{1,21} \subseteq D_1 \dot{+} D_{21}$ and a set $M_{20} \subseteq D_{20}$ such that the following holds:

1. For any fixed $t \in \mathbb{R}_+$, $(x_1 + x_{21}) \in M_{1,21}$ there exists a unique $x_{20} \in M_{20}$ such that
$$(t, x) \in L_0 = \{(t, x) \in \mathbb{R}_+ \times X \mid (t, x) \text{ satisfies equations (7), (8)}\}.$$
2. A function $f(t, x)$ has the continuous (strong) derivative $\partial_x f$ on $\mathbb{R}_+ \times D$. For any fixed $t \in \mathbb{R}_+$, $x \in D$ such that $(x_1 + x_{21}) \in M_{1,21}$, $x_{20} \in M_{20}$ and $(t, x) \in L_0$, the operator
$$W_{t,x} := \Big[(Q_{20}^{(1)} + Q_{21})\partial_x f(t, x) P_{20} - BP_{20}^{(1)}\Big]\Big|_{D_{20}} \colon D_{20} \to Y_{20}^1 \dot{+} Y_{21} \tag{11}$$
has the inverse $W_{t,x}^{-1} \in \mathrm{L}(Y_{20}^1 \dot{+} Y_{21}, D_{20})$.
3. If $M_{1,21} \neq (P_1 + P_{21})X$, then the following holds.
The component $(x_1 + x_{21})(t) = (P_1 + P_{21})x(t)$ of each solution $x(t)$ with the initial point $(t_0, x_0) \in L_0$, for which $(P_1 + P_{21})x_0 \in M_{1,21}$, $P_{20}x_0 \in M_{20}$, can never leave $M_{1,21}$ (i.e., it remains in $M_{1,21}$ for all t from the maximal interval of existence of the solution).
4. If $M_{1,21}$ is unbounded, then the following holds.
There exists a number $R > 0$ (R can be sufficiently large), a function $V \in C^1\big(\mathbb{R}_+ \times M_R, \mathbb{R}\big)$ positive on $\mathbb{R}_+ \times M_R$, where $M_R = \{(x_1, x_{21}) \mid x_1 + x_{21} \in M_{1,21},\ \|x_1 + x_{21}\| > R\}$, and a function $\chi \in C(\mathbb{R}_+ \times (0, \infty), \mathbb{R})$ such that:
   4.1 $\lim\limits_{\|(x_1, x_{21})\| \to +\infty} V(t, x_1, x_{21}) = +\infty$ uniformly in t on each finite interval $[a, b) \subset \mathbb{R}_+$;
   4.2 for each $t \in \mathbb{R}_+$, $(x_1, x_{21}) \in M_R$, $x_{20} \in M_{20}$ such that $(t, x) \in L_0$, the derivative (9) of the function V along the trajectories of equations (5), (6) satisfies the inequality $\dot{V}_{(5),(6)}(t, x_1, x_{21}) \le \chi(t, V(t, x_1, x_{21}))$;
   4.3 the differential inequality $\dot{v} \le \chi(t, v)$ $(t \in \mathbb{R}_+)$ does not have positive solutions with finite escape time.

Then there exists a unique global (i.e., on $[t_0, \infty)$) solution of IVP (1), (10) for each initial point $(t_0, x_0) \in L_0$ for which $(P_1 + P_{21})x_0 \in M_{1,21}$ and $P_{20}x_0 \in M_{20}$.

Corollary. If in the conditions of Theorem 1 the sets $M_{1,21}$ and M_{20} are bounded, then equation (1) is uniformly dissipative (uniformly ultimately bounded) for the initial points $(t_0, x_0) \in L_0$ for which $(P_1 + P_{21})x_0 \in M_{1,21}$ and $P_{20}x_0 \in M_{20}$.

Discussions

The local solvability of an abstract DAE of the form (1) for the nonlinear function of a special form has been studied in [1].

In [2–5], the theorems and methods for the study of the global solvability, Lyapunov stability and instability, asymptotic stability, complete stability (asymptotic stability in the large) and the dissipativity of semilinear DAEs with the regular pencil of index not higher than 1 (or the singular pencil whose regular block is a regular pencil of index not higher than 1) in finite-dimensional spaces have been obtained. In the present work, we extend these results to the higher-index abstract DAEs in Banach spaces.

Selected publications

1. Favini, A., Rutkas, A. Existence and uniqueness of solutions of some abstract degenerate nonlinear equations. Diff. Int. Eqs. **12**(3), 373-394 (1999).
2. Filipkovskaya, M.S. Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. I, II. Differ. Equations **57**(1, 2), 19–40, 196–209 (2021).
3. Filipkovska (Filipkovskaya), M. Existence, boundedness and stability of solutions of time-varying semilinear differential-algebraic equations. Global and Stochastic Analysis **7**(2), 169–195 (2020).
4. Filipkovska, M. Criterion of the global solvability of regular and singular differential-algebraic equations. Journal of Mathematical Sciences (2024) [Online publication]. https://doi.org/10.1007/s10958-024-07152-7
5. Filipkovska, M. Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks. Journal of Mathematical Physics, Analysis, Geometry **19**(4), 719-765 (2023).
6. Filipkovska, M.S. Two combined methods for the global solution of implicit semilinear differential equations with the use of spectral projectors and Taylor expansions. Int. J. of Computing Science and Mathematics **15**(1), 1–29 (2022).
7. Keldysh, M.V. On the completeness of the eigenfunctions of some classes of non-self-adjoint linear operators. Russian Mathematical Surveys, 1971, **26**(4), 15–44.
8. Rutkas, A.G., Vlasenko, L.A. Existence of solutions of degenerate nonlinear differential operator equations. Nonlinear Oscillations **4**(2), 252–263 (2001).