Modern calibration strategies for macroscopic traffic flow models

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joint work with Alexandra Würth and Mickaël Binois

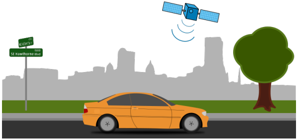
Spatio-temporal evolution of aggregate quantities:

- the (mean) traffic density ρ: number of vehicles per unit space
- the (mean) velocity v: distance covered by vehicles per unit time
- the traffic flow $q = \rho v$: number of vehicles per unit time

Data sources:

@Wikipedia

magnetic loop detectors, video recordings, floating car data, etc



Loop detector data[1]

Example:

355#M3E;	2;	255;	Ma;	01/09/15;	00:00;	1240;	C;	109;	10;
355#M3E;	2;	255;	Ma;	01/09/15;	00:00;	1407;	D;	96;	54;
305#M3e;	2;	255;	Ma;	01/09/15;	00:00;	1449;	D;	100;	10;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	1874;	D;	104;	37;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	2248;	D;	83;	36;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	2368;	D;	78;	39;
577#M6C;	16;	474;	Ma;	01/09/15;	00:00;	4237;	D;	120;	40;
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

where:

loop code

road milestone

distance from the milestone (m)

day of the week

date

time (hours:minutes)

seconds and hundredth

lane

vehicle speed (km/h)

vehicle length (dm)

Sensors types

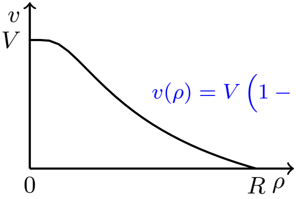
- **single:** vehicle count at fixed position (flow rate) and *occupancy* measure (density and velocity inferred, assuming average vehicle length)
- **double:** direct speed measure

[1] courtesy of DirMED

Lighthill-Whitham-Richards (LWR) model (mid 50s)

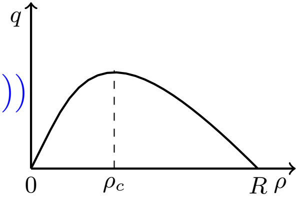
- Mass conservation equation: $\partial_t \rho + \partial_x q = 0$
- Phenomenological speed-density relation: $v = v(\rho)$

fundamental diagram



speed-density

$$v(\rho) = V\left(1 - \exp\left(\frac{C}{V}\left(1 - \frac{R}{\rho}\right)\right)\right)$$



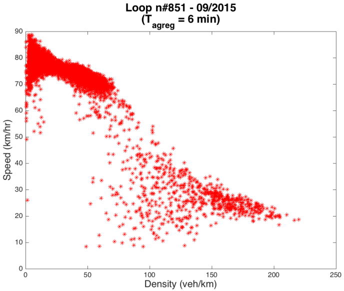
flow-density

R maximal or *jam* density, ρ_c critical density:

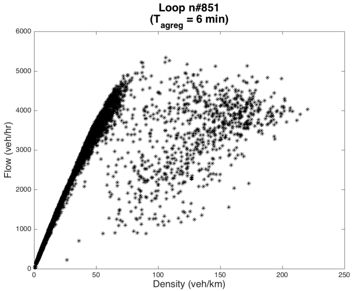
- flux is increasing for $\rho \leq \rho_c$: free-flow phase
- flux is decreasing for $\rho \geq \rho_c$: congestion phase

$v = v(\rho)$?

Experimental observations of fundamental diagrams are more complex than postulated by first order traffic models



speed-density



flow-density

Data from A51 (Marseilles, France), September 2015, courtesy of DirMED

Improved models

- **Additional equation for v: second order models**
  [Payne (1971); Helbing (1996); Aw-Rascle (2000); Zhang (2002); Colombo (2002); Lebacque-et-al (2005); ...]

- **Phase transition**
  [Colombo (2002); Goatin (2006); Blandin-et-al (2011); ...]

- **Stochastic and averaged models**
  [Boel-Mihaylova (2006); Wang-Ni-Chen-Li (2010); Chen-Wang (2011); Sumalee&al (2011); Jabari-Liu (2011); ...]

- **Multi-class**
  [Wong-Wong (2002); Benzoni-Colombo (2003); VanLint-Hoogendorn-Schreurer (2008); Nair-Mahmassani-Hooks (2011); Fan-Work (2015); Gashaw-Goatin-Harri (2016); ...]

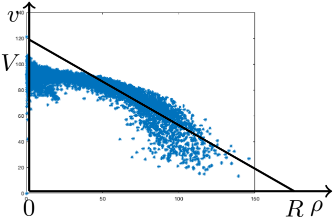
- **Multi-lane**
  [Michalopoulos-Beskos-Yamauchi (1984); Klar-Wagener (1999); Greenberg-Klar-Rascle (2003); Colombo-Corli (2006); HoldenRisebro (2019); ...]

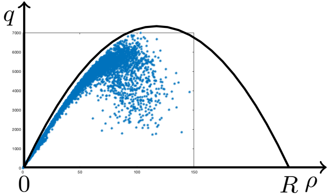
- **Non-local models**
  [Blandin-Goatin (2016); Friedrich-Kolb-Göttlich (2018); ...]

Generalized Second Order Model[2] (GSOM)

- Conservation of vehicles: $\partial_t \rho + \partial_x(\rho v) = 0$
- Variable fundamental diagram: $v = \mathcal{V}(\rho, w)$, w driver attribute
- Transport equation for w (follows vehicle trajectories):

$$\partial_t w + v \partial_x w = 0$$





Remarks

- $w = const \quad \longrightarrow \quad$ LWR
- $\mathcal{V}(\rho, w) = w - p(\rho) \quad \longrightarrow \quad$ ARZ
- $\mathcal{V}(\rho, w) = w\left(1 - \exp\left(\frac{C}{V}\left(1 - \frac{R}{\rho}\right)\right)\right)$

[2] [Lebacque-Mammar-Salem, Transportation and Traffic Theory (2007)]

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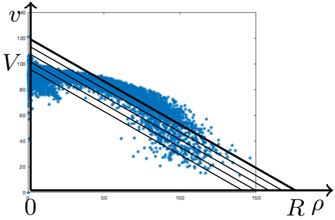

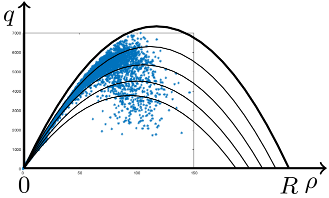

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Initial-Boundary Value Problem for GSOM[3]

$$
\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0 \end{cases} \qquad x \in]x_{in}, x_{out}[\,, \ t > 0
$$

$$
(\rho, w)(0, x) = (\rho_0, w_0)(x) \qquad x \in]x_{in}, x_{out}[
$$

$$
(\rho, w)(t, x_{in}) = (\rho_{in}, w_{in})(t) \qquad t > 0
$$

$$
(\rho, w)(t, x_{out}) = (\rho_{out}, w_{out})(t) \qquad t > 0
$$

where

$$
\begin{aligned}
&\mathcal{V}(\rho, w) \geq 0, \quad \mathcal{V}(0, w) = w, \\
&2\mathcal{V}_\rho(\rho, w) + \rho \mathcal{V}_{\rho\rho}(\rho, w) < 0 \text{ for } w > 0, \\
&\mathcal{V}_w(\rho, w) > 0, \\
&\forall w > 0 \quad \exists\ R(w) > 0 : \quad \mathcal{V}(R(w), w) = 0,
\end{aligned}
$$

such that

$$
\lambda_1(\rho, w) = \mathcal{V}(\rho, w) + \rho \mathcal{V}_\rho(\rho, w), \qquad \lambda_2(\rho, w) = \mathcal{V}(\rho, w) > 0
$$

$\Longrightarrow$ existence of entropy weak solutions for BV data

[3] [Goatin-Würth, Nonlinear Analysis (2023)]

Numerical approximation[4]

Hilliges-Weidlich (HW) finite volume scheme: $\mathbf{u} = (\rho, y)^T = (\rho, \rho w)^T$

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n\right), \qquad \mathbf{F}_{j+1/2}^n = (F_{j+1/2}^{\rho,n}, F_{j+1/2}^{y,n})^T$$

where

$$F_{j+1/2}^{\rho,n} = \rho_j^n\,\mathcal{V}^+(\rho_{j+1}^n, w_{j+1}^n) \qquad \text{and} \qquad F_{j+1/2}^{y,n} = w_j^n\,F_{j+1/2}^{\rho,n}$$

and

$$(CFL) \quad \Delta t\left\{\|\mathcal{V}\|_\infty + R(w_{max})\|\mathcal{V}_\rho\|_\infty\right\} \leq \Delta x,$$

Maximum principle

- If $\Delta t\,\|\mathcal{V}\|_\infty \leq \Delta x$, scheme (HW) is positivity preserving.
- Under (CFL), if $R(w) = \mathbf{R}_{\max}\ \ \forall w$, the approximate solution constructed by (HW) scheme satisfies $\rho_j^n \leq \mathbf{R}_{\max}$ (and $\mathcal{V}(\rho_j^n, w_j^n) \geq 0$) for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

$\sim 40\%$ computational time reduction

[4] [Würth-Goatin-Villada, HYP2022 Proceedings]

Bayesian calibration with bias[5]

Combining field data and computer simulations for calibration and prediction

[Kennedy-O'Hagan, J. Royal Statistical Society B, 2001; Higdon&al, SISC, 2004]

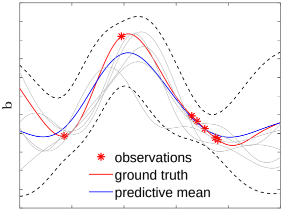
Look for θ^*, b and ϵ s.t.

$$\underbrace{y^F(t,x)}_{\text{observation}} = \underbrace{y^M(t,x,\theta^*)}_{\text{simulation}} + \underbrace{b(t,x,\theta^*)}_{\text{discrepancy}} + \underbrace{\epsilon(t,x)}_{\text{obs. error}}$$

assuming $b \sim \mathcal{MVN}(0, \sigma^2 Corr)$ and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$ $\longrightarrow$ Gaussian Process

Correct the simulation output as

$$y_c^M(t,x,\theta^*) = y^M(t,x,\theta^*) + m_N(t,x).$$



[5] [Würth-Binois-Goatin-Göttlich, Adv. Comput. Math., 2022]

Gaussian process modeling[6]

- Set of N observation points: $\mathcal{X}_N = \big((t_1, x_1), \ldots, (t_N, x_N)\big)$
- Set of observed (noisy) biases: $\mathbf{b}_N(\theta) = y^F(\mathcal{X}_N) - y^M(\mathcal{X}_N, \theta)$

[6] [Rasmussen-Williams, *Gaussian processes for machine learning*, 2006]

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- Set of observed (noisy) biases: $\mathbf{b}_N(\theta) = y^F(\mathcal{X}_N) - y^M(\mathcal{X}_N, \theta)$
- $\mathbf{b}_N$ is modeled by a **realization** of a **MVN distribution**:

$$\mathbf{b}_N \sim \mathcal{N}(\mathbf{0}_N, \mathbf{K}_N) \text{ with } \mathbf{K}_N = \sigma^2(\mathbf{C}_N + g\mathbf{I}_N), g = \frac{\sigma_\varepsilon^2}{\sigma^2}$$

and (stationary) **Gaussian kernel** correlation matrix:

$$\mathbf{C}_N := \mathbb{C}orr\big(b(t,x), b(t',x')\big) = \exp\left(-\frac{(t-t')^2}{{l_1}^2}\right) \exp\left(-\frac{(x-x')^2}{{l_2}^2}\right)$$

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- **(Hyper)parameter estimation** by maximization of the concentrated log-likelihood:

$$\log \tilde{\mathcal{L}}(l_1, l_2, g, \mathbf{b}_N) = -\frac{N}{2}\log 2\pi - \frac{N}{2}\log \hat{\sigma}^2(l_1, l_2, g, \mathbf{b}_N) - \frac{1}{2}\log|\mathbf{C}_N + g\mathbf{I}_N| - \frac{N}{2}$$

$$\left(\text{with} \quad \hat{\sigma}^2(l_1, l_2, g, \mathbf{b}_N) = \frac{\mathbf{b}_N^\top(\mathbf{C}_N + g\mathbf{I}_N)^{-1}\mathbf{b}_N}{n}\right)$$

$$\boxed{\theta^* \leftarrow \underset{\theta}{\operatorname{argmax}} \left[\max_{l_1, l_2, g} \log \tilde{\mathcal{L}}\left(l_1, l_2, g, \mathbf{b}_N(\theta)\right)\right]}$$

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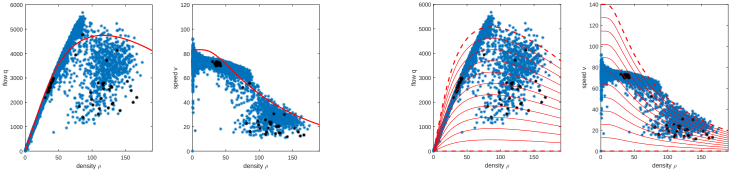
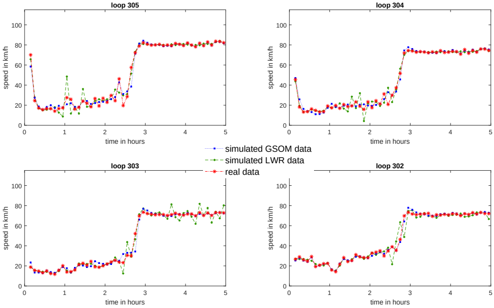
Comparing LWR and GSOM



Optimal parameters - fundamental diagram (DirMED data)

	LWR			GSOM		
	V^*	C^*	R^*	V^*	C^*	R^*
DirMed	83.51	23.84	390.00	94.74	17.02	348.22
RTMC	87.37	29.15	422.51	91.56	18.39	478.23

Comparing LWR and GSOM



Optimal parameters - speed profile validation (DirMED data)

	LWR				GSOM			
	E^{flow}	E^{speed}	E^{density}	**E**	E^{flow}	E^{speed}	E^{density}	**E**
DirMed	1.245	1.370	3.695	6.311 (+85%)	**0.917**	**0.879**	**1.619**	**3.414**
RTMC	**0.883**	1.929	2.548	5.360 (+93%)	1.184	**0.863**	**0.730**	**2.777**

Bayesian formulation (MCMC method, Metropolis algorithm):

- prior multivariate normal distribution

$$\pi(\theta_{\mathrm{VCR}}) \propto \frac{1}{\sqrt{|\Sigma_{\theta_{\mathrm{VCR}}}|}} \exp\left\{-0.5\left(\theta_{\mathrm{VCR}} - \mu_{\theta_{\mathrm{VCR}}}\right)^{\top} \Sigma^{-1}_{\theta_{\mathrm{VCR}}} \left(\theta_{\mathrm{VCR}} - \mu_{\theta_{\mathrm{VCR}}}\right)\right\}$$

  with mean $\mu_{\theta_{\mathrm{VCR}}}$ and covariance matrix $\Sigma_{\theta_{\mathrm{VCR}}}$.
- proposal multivariate normal distribution

$$\hat{\theta}_{\mathrm{VCR}} \sim \mathcal{N}((\theta_{\mathrm{VCR}})_{i-1}, \Sigma^{p}_{\mathrm{VCR}})$$

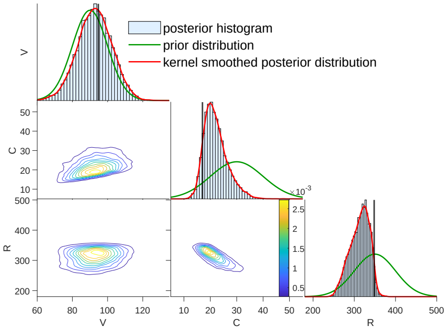
  with covariance matrix $\Sigma^{p}_{\mathrm{VCR}}$.
- sampling model for y^F

$$\mathcal{L}(y^F \mid \theta^*) \propto \left|\hat{\sigma}^2(\theta^*)\left(\mathbf{C}_n\left(l_1, l_2\right) + g\mathbf{I}_n\right)\right|^{-1/2}.$$

- posterior distribution (Bayes theorem)

$$\boxed{\pi(\theta^* \mid y^F) = \frac{\mathcal{L}\left(y^F \mid \theta^*\right) \times \pi(\theta_{\mathrm{VCR}})}{\pi(y^F)}}$$

Metropolis algorithm: 10^5 iterations, 10% *burn-in phase*
effective sample sizes: 720 (LWR), 11250 (GSOM)



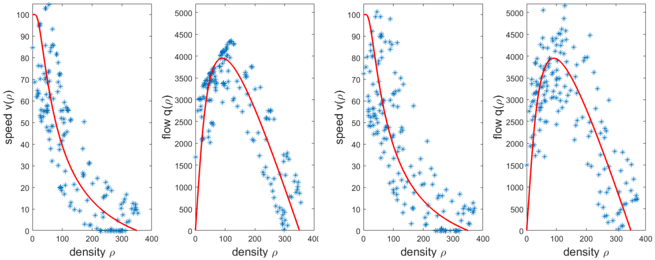
GSOM model - parameters distribution (DirMED data)

	LWR				GSOM			
	$E^{\text{flow}}_{\text{MCMC}}$	$E^{\text{speed}}_{\text{MCMC}}$	$E^{\text{density}}_{\text{MCMC}}$	$\mathbf{E}_{\text{MCMC}}$	$E^{\text{flow}}_{\text{MCMC}}$	$E^{\text{speed}}_{\text{MCMC}}$	$E^{\text{density}}_{\text{MCMC}}$	$\mathbf{E}_{\text{MCMC}}$
DirMed	1.056	1.272	4.336	6.664 (+101%)	**0.929**	**0.827**	**1.567**	**3.322**
RTMC	**0.896**	2.507	2.218	5.621 (+111%)	1.166	**0.687**	**0.813**	**2.667**

Comparison of calibration approaches[7]

On synthetic data with discrepancy and noise:

- Run a simulation y_{sim} with given $\theta^0 = (V^0, C^0, R^0) = (100, 20, 350)$
- Add bias to y_{sim} ($\tau \in \{0, 0.02, 0.1\}$):
  read data: $y^P(t,x) := y_{\text{sim}}(t,x) + \tau \max(y_{\text{sim}}(t,x)) \cdot \sin(t+x)$
- Generate 6 min noisy average loop detector data ($s = 0.15$):
  field data: $y^F = \bar{y}^P + s\bar{y}^P \mathcal{N}(0,1)$

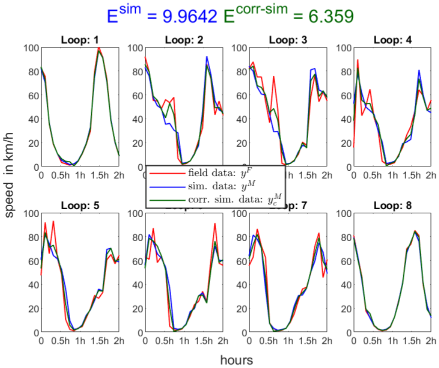


Fundamental diagrams for 6 min average real (y^P) and noisy data (y^F), $\tau = 0.1$.

[7] [Würth-Binois-Goatin, MT-ITS 2023]

Comparison of calibration approaches

Speed profiles (GSOM - KOH) and RSME between $\bar{y}^P$ and corrected simulation:



	τ	LWR	GSOM
L2	0.00	**2.44**	2.89
	0.02	4.85	6.65
	0.10	6.29	9.40

	τ	LWR	GSOM
KOH	0.00	2.48	3.33
	0.02	**4.75**	6.62
	0.10	**6.11**	7.66

Conclusion and perspectives

Summary

- **Existence** of entropy weak solutions for **IBVP with vacuum**
- **Cheap** finite volume scheme
- Statistical framework for **traffic state reconstruction**
- Introduction of a **bias term** to compensate model limitations
- Extension to **traffic state** and **travel time prediction**
  [Würth-Binois-Goatin, *Traffic prediction by combining macroscopic models and Gaussian processes*, submitted.]

Perspectives

- Consideration of more complex traffic scenarios.
- Local variations of the bias on the road.

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