Optimal control of nonlocal conservation laws and the singular limit

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Alexander Keimer[1], Lukas Pflug, Jakob Rodestock

Institute of Mathematics, University of Rostock, Germany

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[1] alexander.keimer@uni-rostock.de

Why **Nonlocal** and what does it even mean?

Figure 1: Star Trek: Picard, Episode 2 ("Maps and Legends"), aired on January 30th, 2020.

The dynamics considered

Problem Setup: Nonlocal conservation laws

Consider

$$q_t(t,x) + \partial_x\Big(V\big(W[q,\gamma](t,x)\big)q(t,x)\Big) = 0 \qquad (t,x) \in (0,T)\times\mathbb{R}$$

$$q(0,x) = q_0(x) \qquad x \in \mathbb{R}$$

supplemented by the nonlocal term W, averaging the "density" in space "downstream"

$$W[q,\gamma](t,x) := \int_x^\infty \gamma(x-y)q(t,y)\,\mathrm{d}y, \qquad (t,x) \in (0,T)\times\mathbb{R},$$

with

1. $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}): \ V' \leqq 0$
2. $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$
3. $\gamma \in BV(\mathbb{R}_{<0})$ monotonically increasing.

Remark: Applications

- Traffic (velocity depends on the density downstream)
- Pedestrian flow, herd dynamics, swarm behavior, chemical ripening processes, etc....

Well-posedness: Existence and Uniqueness

Theorem (Existence & Uniqueness, stability)

Given the previously stated assumptions, the nonlocal conservation law admits a **unique weak** *solution*

$$q \in C\big([0,T]; L^1_{loc}(\mathbb{R})\big) \cap L^\infty((0,T); L^\infty(\mathbb{R}))$$

satisfying the following maximum principle

$$\operatorname*{ess\,inf}_{y\in\mathbb{R}} q_0(y) \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})}, \qquad (t,x) \in (0,T) \times \mathbb{R} \text{ a.e.}$$

Even more, for $q_0 \in C^1(\mathbb{R})$, we have that q is even a strong solution.

Remark: Stability and solution properties

- Any weak solution can be approximated by strong (and even classical) solutions in $L^1(\mathbb{R})$, if $q_0 \in TV(\mathbb{R})$.
- No fully local behavior anymore, i.e., solution has to be known between x and ∞ to "advance" in time.
- Still finite propagation of mass, but infinite speed of "information".
- No entropy condition for uniqueness required.

Singular limit problem: Convergence to local entropy solution when $\gamma \rightharpoonup \delta$?

For $\eta \in \mathbb{R}_{>0}$ let $q_\eta \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$ denote the **weak** solution of

$$\begin{aligned} q_t(t,x) &= -\partial_x \left(V\big(W[q,\gamma_\eta](t,x)\big)q(t,x)\right), && (t,x) \in (0,T)\times\mathbb{R},\\ q(0,x) &= q_0(x), && x \in \mathbb{R},\\ W[q,\gamma_\eta](t,x) &:= \tfrac{1}{\eta}\int_x^\infty \gamma\big(\tfrac{x-y}{\eta}\big)q(t,y)\,\mathrm{d}y, && (t,x)\in(0,T)\times\mathbb{R}. \end{aligned}$$

Denote by $q^* \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$ the **weak entropy** solution to the local counter-part conservation law (the “LWR” model in traffic)

$$\begin{aligned} \partial_t q(t,x) &= -\partial_x\big(V(q(t,x))q(t,x)\big), && (t,x)\in(0,T)\times\mathbb{R},\\ q(0,x) &= q_0(x), && x\in\mathbb{R}. \end{aligned}$$

Singular limit (I)

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Do we have $q_\eta \overset{\eta \to 0}{\longrightarrow} q^*$?

Singular limit (II)

Theorem: Convergence against the local entropy solution

Given **reasonable** kernels γ and $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap TV(\mathbb{R})$. Then, it holds

$$\lim_{\eta\to 0} \|W[q_\eta, \gamma_\eta - q^*\|_{C([0,T];L^1(\mathbb{R}))} = 0 = \lim_{\eta\to 0} \|q_\eta - q^*\|_{C([0,T];L^1(\mathbb{R}))},$$

where q^* is the (local) entropy solution introduced before.

Remark

The singular limit convergence in this framework is by now quite well understood.

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$$\Longrightarrow$$ **Optimal control** ☺

Barely any results available, neither existence of minimizers, nor numerical methods.

Optimal Control (I)

Problem setup: Optimal control of nonlocal conservation laws

Consider

$$\inf_{\substack{q_0 \in \mathcal{Q}\\ V \in \mathcal{V}\\ \gamma \in \mathcal{G}}} \quad \|q_\eta - q_{\mathsf{d}}\|^2_{L^2((0,T)\times\mathbb{R})} + \|q_\eta(T,\cdot) - q_T\|^2_{L^2(\mathbb{R})} + \|q_0 - q_{0,\mathsf{d}}\|^2_{L^2(\mathbb{R})}$$

$$\text{s.t.}$$

$$\begin{aligned}
\partial_t q_\eta(t,x) &= -\partial_x\big(V\left(W[q_\eta,\gamma_\eta](t,x)\right) q_\eta(t,x)\big), && (t,x)\in(0,T)\times\mathbb{R},\\
W[q_\eta,\gamma_\eta](t,x) &= \tfrac{1}{\eta}\int_x^\infty \gamma\big(\tfrac{x-y}{\eta}\big) q_\eta(t,y)\,\mathrm{d}y, && (t,x)\in(0,T)\times\mathbb{R},\\
q_\eta(0,x) &= q_0(x), && x\in\mathbb{R},
\end{aligned}$$

with $q_{\mathsf{d}} \in L^2((0,T)\times\mathbb{R})$ and $q_T, q_{0,\mathsf{d}} \in L^2(\mathbb{R})$.
Does there exist a minimizer? What do we require on $\mathcal{Q}$, $\mathcal{V}$, $\mathcal{G}$?

Remark: Different objectives

- The objective function can be generalized to lower weakly semi-continuous functionals.
- Instead of tracking q_η it could be advantageous to track the nonlocal operator $W[q_\eta,\gamma_\eta]$ and the theory applies then as well.

Optimal Control (II)

Theorem: Existence of a minimizer

Let $q_{\max} \in \mathbb{R}_{>0}$, $C_{\mathcal{V}} \in \mathbb{R}_{>0}$ and $C_{\mathcal{G}} \in \mathbb{R}_{>0}, \varepsilon \in \mathbb{R}_{>0},\ \bar{R} \in \mathbb{R}_{<0}$. Define

- $\mathcal{Q} := \{q_0 \in L^2(\mathbb{R}) : 0 \leq q_0 \leq q_{\max} \text{ a.e.}\}$
- $\mathcal{V} := \Big\{V \in W^{1,\infty}(\mathbb{R}; \mathbb{R}_{\geq 0}) : V' \leq 0 \text{ a.e. on } [0, q_{\max}],\ \|V\|_{W^{1,\infty}([0,q_{\max}])} \leq C_{\mathcal{V}}\Big\}$
- $\mathcal{G} := \Big\{\gamma \in BV\big(\mathbb{R}_{<0}; \mathbb{R}_{\geq 0}\big) : \gamma \text{ increasing},\ \|\gamma\|_{L^\infty(\mathbb{R})} \leq C_{\mathcal{G}},$
  $\|\gamma\|_{L^1(\mathbb{R}_{<0})} = 1,\ \gamma(x) \leq \frac{1}{(-x)^{1+\varepsilon}} \text{ for a.e. } x \in \mathbb{R}_{<\bar{R}}\Big\}.$

Then, there exists a minimizer $(q^*, V^*, \gamma^*) \in \mathcal{Q} \times \mathcal{V} \times \mathcal{G}$ solving the previously stated optimal control problem.

Remark: Assumptions on γ, Uniqueness

- If we assume that γ is compactly supported, we do not require the growth condition "close to infinity."
- We cannot expect uniqueness of minimizers.

Outline of proof

- Choose sequence $\big(q_{0,k}, V_k, \gamma_k\big) \subseteq \mathcal{Q} \times \mathcal{V} \times \mathcal{G}$ such that

$$\lim_{k\to\infty} J\big(q_{0,k}, q_\eta[q_{0,k}, V_k, \gamma_k]\big) = \inf_{q_0\in\mathcal{Q}, V\in\mathcal{V}, \gamma\in\mathcal{G}} J\big(q_0, q_\eta[q_0, V, \gamma]\big).$$

- Along a subsequence:
  - $q_{0,k} \rightharpoonup q_0^* \in \mathcal{Q}$ in $L^2(\mathbb{R})$ (Banach-Alaoglu)
  - $V_k \to V^* \in \mathcal{V}$ uniformly on $\mathbb{R}$ (Arzelà-Ascoli)
  - $\gamma_k \to \gamma^* \in \mathcal{G}$ in $L^1(\mathbb{R})$ (Riesz-Kolmogorov)
- Using the characteristics formula for the solution it holds along that subsequence:

$$q_\eta[q_{0,k}, V_k, \gamma_k] \rightharpoonup q_\eta[q_0^*, V^*, \gamma^*] \text{ in } L^2(\mathbb{R})$$

- Lower weakly semi-continuity yields

$$\begin{aligned}\inf_{q_0\in\mathcal{Q}, V\in\mathcal{V}, \gamma\in\mathcal{G}} J\big(q_0, q_\eta[q_0, V, \gamma]\big) &\leq J\big(q_0^*, q_\eta[q_0^*, V^*, \gamma^*]\big)\\ &\overset{\text{l.w.s.c.}}{\leq} \lim_{k\to\infty} J\big(q_{0,k}, q_\eta[q_{0,k}, V_k, \gamma_k]\big)\\ &= \inf_{q_0\in\mathcal{Q}, V\in\mathcal{V}, \gamma\in\mathcal{G}} J\big(q_0, q_\eta[q_0, V, \gamma]\big).\end{aligned}$$

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$\implies (q_0^*, V^*, \gamma^*)$ is a minimizer.

Problem setup: The singular limit for optimal control problems

Consider for $\eta \in \mathbb{R}_{>0}$

$$\big(q_{0,\eta}^{\text{opt}}, V_\eta^{\text{opt}}\big) \in \operatorname*{arg\text{-}min}_{\substack{q_0 \in \mathcal{Q}\\ V \in \mathcal{V}}} \int_{\mathbb{R}} \bigg(\tfrac{1}{\eta} \int_x^\infty \exp\big(\tfrac{x-y}{\eta}\big)\big(q_\eta(T,y) - q_T(y)\big)\,\mathrm{d}y\bigg)^2 \mathrm{d}x$$

$$\text{s.t.}$$

$$\begin{aligned} \partial_t q_\eta(t,x) &= -\partial_x\big(V\left(W[q_\eta,\gamma_\eta](t,x)\right) q_\eta(t,x)\big), && (t,x) \in (0,T)\times\mathbb{R},\\ W[q_\eta,\gamma_\eta](t,x) &= \tfrac{1}{\eta}\int_x^\infty \exp\left(\tfrac{x-y}{\eta}\right) q_\eta(t,y)\,\mathrm{d}y, && (t,x) \in (0,T)\times\mathbb{R},\\ q_\eta(0,x) &= q_0(x), && x \in \mathbb{R}. \end{aligned}$$

The singular limit for optimal control problems (I)

Problem setup: The singular limit for optimal control problems

Consider for $\eta \in \mathbb{R}_{>0}$

$$\begin{aligned}
\big(q_{0,\eta}^{\text{opt}}, V_\eta^{\text{opt}}\big) \in \operatorname*{arg\text{-}min}_{\substack{q_0 \in \mathcal{Q}\\ V \in \mathcal{V}}} & \int_{\mathbb{R}} \left(\tfrac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta})\big(q_\eta(T,y) - q_T(y)\big)\,\mathrm{d}y\right)^2 \mathrm{d}x\\
\text{s.t.}&\\
\partial_t q_\eta(t,x) &= -\partial_x\big(V\left(W[q_\eta,\gamma_\eta](t,x)\right) q_\eta(t,x)\big), && (t,x) \in (0,T)\times\mathbb{R},\\
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q_\eta(0,x) &= q_0(x), && x \in \mathbb{R}.
\end{aligned}$$

Does $\big(q_{0,\eta}^{\text{opt}}, V_\eta^{\text{opt}}\big) \in \mathcal{Q}\times\mathcal{V}$ converge to a solution of the corresponding **local optimal control problem** for $\eta \to 0$

$$\begin{aligned}
\min_{\substack{q_0 \in \mathcal{Q}\\ V \in \mathcal{V}}} & \int_{\mathbb{R}} \big(q^*(T,x) - q_T(x)\big)^2 \,\mathrm{d}x\\
\text{s.t.}&\\
\partial_t q^*(t,x) &= -\partial_x\big(V\left(q^*(t,x)\right) q^*(t,x)\big), && (t,x) \in (0,T)\times\mathbb{R},\\
q^*(0,x) &= q_0(x), && x \in \mathbb{R}?
\end{aligned}$$

The singular limit for optimal control problems (II)

Theorem: Convergence of minimizers

For any subsequence $\eta \to 0$ with

$$q_{0,\eta}^{\text{opt}} \to q_0^{\text{opt},*} \text{ in } L^2(\mathbb{R}),\ V_\eta^{\text{opt}} \to V^{*,\text{opt}} \text{ in } L^\infty$$

the accumulation points $\left(q_0^{\text{opt},*},\ V_\eta^{\text{opt}}\right) \in \mathcal{Q} \times \mathcal{V}$ are solutions to the previously stated local optimal control problem provided that we assume on $\mathcal{Q}$ in addition

$$\exists C_{\text{TV}} :\ |q_0|_{TV(\mathbb{R})} \leq C_{\text{TV}}.$$

Remark: Interpretation and generalizations

- We can approximate optimal control problems subject to local conservation laws with their nonlocal counterparts.
- This is only a first step and work in progress...

Open problems & next steps

- Differentiability of the control to state maps (potentially looking at the nonlocal term)
- Study of the well-posedness of the adjoint equations
- Numerical methods and convergence in the singular limit case
- Model calibration on traffic data sets (Berkeley Circles data set, etc.). Can we showcase that nonlocal traffic models fit “reality” better than the corresponding local ones?
- Optimal control when using Oleinik estimates on the nonlocal conservation law (more restrictive)
- Local vs. nonlocal optimal control of conservation laws
- Optimal control of conservation laws with nonlocal velocity

$$\partial_t q + \partial_x \Big(W[V(q), \gamma](t,x) q(t,x) \Big) = 0$$

- Other nonlocalities with smooth kernels on $\mathbb{R}$.
- ...

Used and related literature (I)

Existence & Uniqueness of nonlocal conservation laws

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- G. M. Coclite, M. Colombo, G. Crippa, N. De Nitti, A. Keimer, E. Marconi, L. Pflug, L. V. Spinolo, *Oleinik-type estimates for nonlocal conservation laws and applications to the nonlocal-to-local limit*, 2024
- . . .

The End?

Thank you very much!