REPLICATOR DYNAMICS AS THE LARGE-POPULATION LIMIT OF A MULTI-STRATEGY DISCRETE MORAN PROCESS

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M. Morandotti, G. Orlando. Preprint 2025.

28.04.2025 – 30.04.2025

#MLPDES25 Workshop, Erlangen (Germany)

THE REPLICATOR EQUATION

THE EQUATION

$$\boxed{\begin{cases} \dot{\lambda}_i = \lambda_i \left((A\lambda)_i - \lambda^T A \lambda \right) & \text{for } i = 1,2,\dots,M \\ + \text{ initial conditions} \end{cases}}$$

- *Unknown:* $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} : [0,T] \longrightarrow \mathbb{R}^M$

- *Parameters:* $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{pmatrix} \in \mathbb{R}^{M \times M}$

A SIMPLE OBSERVATION

$$\dot{\lambda}_i = \lambda_i \left((A\lambda)_i - \lambda^T A \lambda \right)$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\sum_{i=1}^{M} \lambda_i \right) = \sum_{i=1}^{M} \lambda_i (A\lambda)_i - \left(\sum_{i=1}^{M} \lambda_i \right) \lambda^T A \lambda$$

$$= \lambda^T A \lambda - \left(\sum_{i=1}^{M} \lambda_i \right) \lambda^T A \lambda =$$

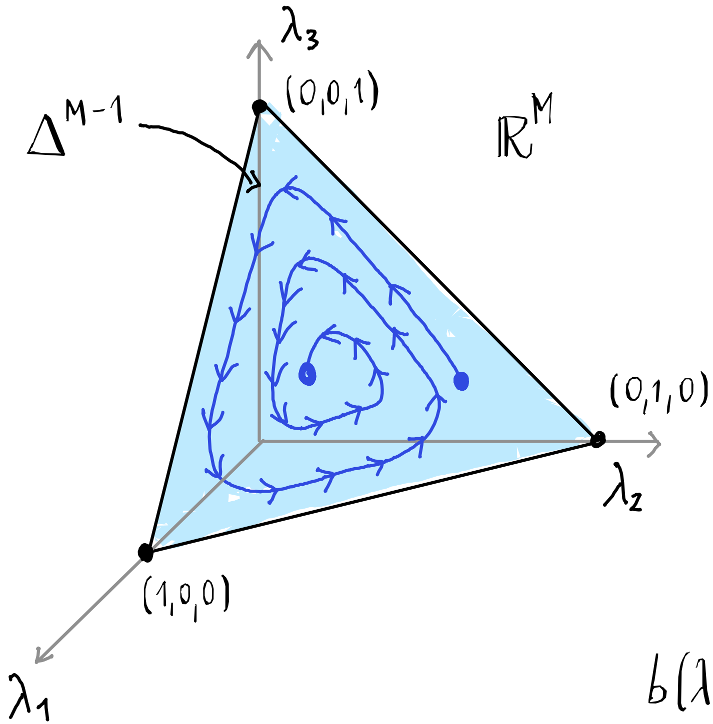
$$= \left(1 - \sum_{i=1}^{M} \lambda_i \right) \lambda^T A \lambda$$

$$\Downarrow$$

If $\sum_{i=1}^{M} \lambda_i = 1$ at time $t = 0$,

then $\sum_{i=1}^{M} \lambda_i = 1$ for all times.

AN EQUATION ON THE SIMPLEX Δ^{M-1}



$$\dot{\lambda}_i = \lambda_i \left((A\lambda)_i - \lambda^T A \lambda \right)$$

$$\Updownarrow$$

$$\boxed{\dot{\lambda} = b(\lambda)}$$

with $b : \Delta^{M-1} \longrightarrow T\Delta^{M-1}$

$$b(\lambda) = \begin{pmatrix} b_1(\lambda) \\ \vdots \\ b_M(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda_1\left((A\lambda)_1 - \lambda^T A \lambda\right) \\ \vdots \\ \lambda_M\left((A\lambda)_M - \lambda^T A \lambda\right) \end{pmatrix}$$

(note: $\sum_{i=1}^{M} b_i(\lambda) = 0$)

$$\Delta^{M-1} = \left\{ \lambda \in \mathbb{R}^M : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{M} \lambda_i = 1 \right\}$$

IT'S AN EVOLUTIONARY GAME

Strategies: $\mathcal{U} = \{u_1, u_2, \dots, u_M\}$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} : [0,T] \longrightarrow \Delta^{M-1}.$$

$$A = \begin{array}{c|cccc} & u_1 & u_2 & \cdots & u_M \\ \hline u_1 & a_{11} & a_{12} & \cdots & a_{1M} \\ u_2 & a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_M & a_{M1} & a_{M2} & \cdots & a_{MM} \end{array}$$

REPLICATOR EQUATION:

$$\dot{\lambda}_i = \lambda_i \big(\underbrace{(A\lambda)_i} - \underbrace{\lambda^T A \lambda} \big)$$

$$(A\lambda)_i = \sum_{j=1}^{M} a_{ij} \lambda_j$$

$$\lambda^T A \lambda = \sum_{i=1}^{M} \lambda_i (A\lambda)_i$$

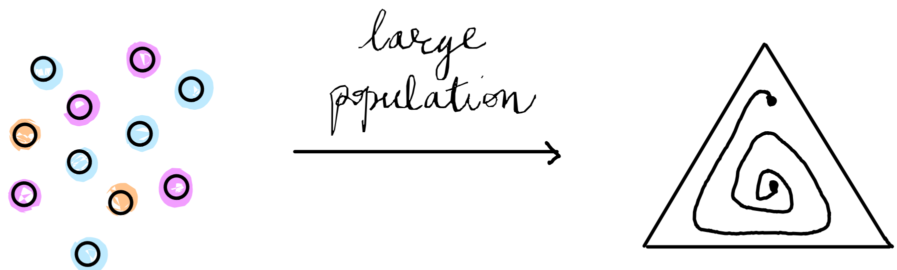
INTERPRETATION: u_i tends to replicate if its expected payoff is better than the average one.

QUESTION

Can we derive rigorously the

REPLICATOR EQUATION

from a finite population game?



ANSWER: yes.

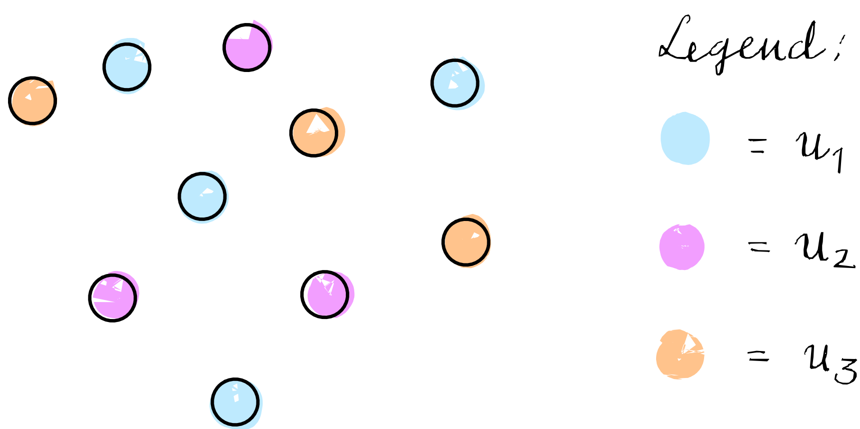
THE (UNIFORM) MORAN PROCESS

POPULATION: N agents $\{1, 2, \dots, N\}$, n: agent

STRATEGIES: M strategies $\mathcal{U} = \{u_1, u_2, \dots, u_M\}$

PROPORTIONS: $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_M)^T \in \Delta^{M-1}$

$$\lambda_i = \frac{\#\{\text{agents with strategy } u_i\}}{N}$$



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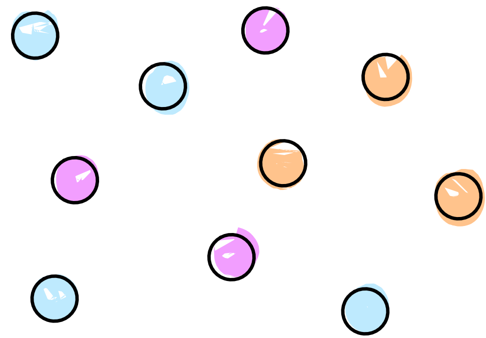


$t = t_h$

BIRTH-DEATH PROCESS:

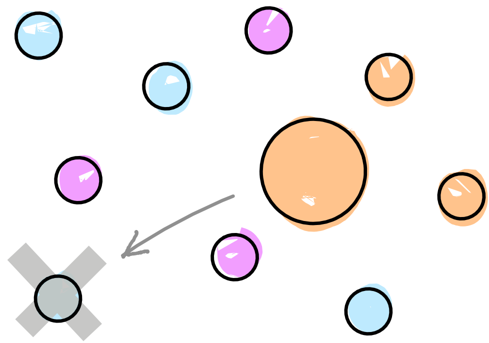
- An agent is selected randomly uniformly to repliate its strategy:

$$\mathbb{P}(n \text{ selected for replication}) = \frac{1}{N}$$

- An agent is selected randomly uniformly to abandon its strategy:

$$\mathbb{P}(n' \text{ selected to die}) = \frac{1}{N}$$

THE (UNIFORM) MORAN PROCESS



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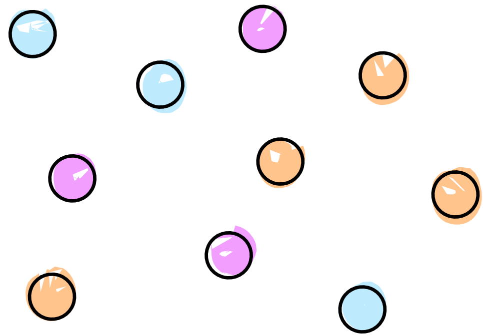

$t = t_{h+1}$

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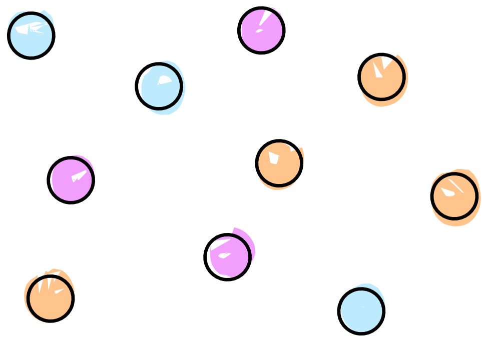


DISCRETE MARKOV CHAIN:

State at step t_n:

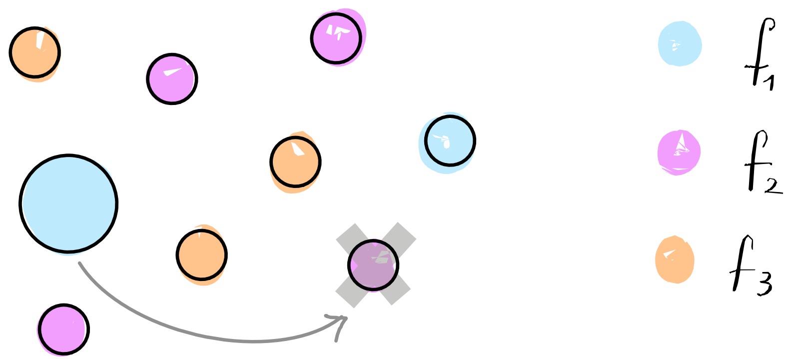
$$(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_M) = \lambda$$

State at time t_{n+1}:

$$(\lambda_1, \dots, \lambda_i + \tfrac{1}{N}, \dots, \lambda_j - \tfrac{1}{N}, \dots, \lambda_M) = \lambda + \tfrac{1}{N}(e_i - e_j)$$

with probability $\lambda_i \lambda_j$

FITNESS IN A MORAN PROCESS



BIRTH-DEATH PROCESS:

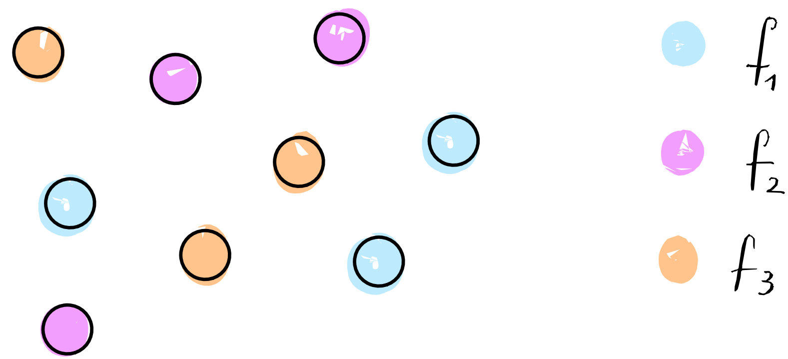
- $\mathbb{P}$(n with u_i selected for replication) $= \dfrac{\frac{1}{N} f_i}{\sum_{j=1}^{M} \lambda_j f_j}$

  $\mathbb{P}$(u_i replicates) $= \dfrac{\lambda_i f_i}{\sum_{\ell=1}^{M} \lambda_\ell f_\ell}$

  Note: if all $f_i = 1$, then it is uniform

- death as before (uniform)

FITNESS IN A MORAN PROCESS



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with probability $\dfrac{\lambda_i f_i \lambda_j}{\sum_{l=1}^{M} \lambda_l f_l}$

PAYOFFS IN A MORAN PROCESS

	u_1	u_2	$\cdots$	u_M
u_1	a_{11}	a_{12}	$\cdots$	a_{1M}
u_2	a_{21}	a_{22}	$\cdots$	a_{2M}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
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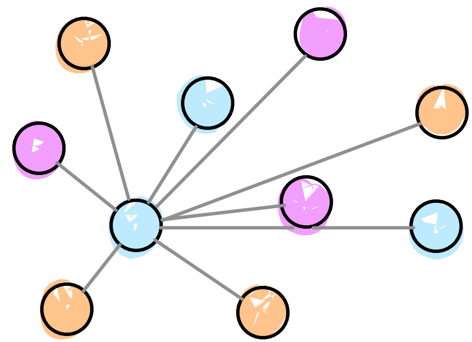

An agent n with strategy u_i has an expected payoff

$$\pi_i = \sum_{\substack{j=1 \\ j \neq i}}^{M} a_{ij} \frac{\#\{\text{w/ strategy } u_j\}}{N-1} + a_{ii} \frac{\#\{\text{w/ strategy } u_i\} - 1}{N-1}$$

$$\approx (A\lambda)_i$$

for $N \gg 1$

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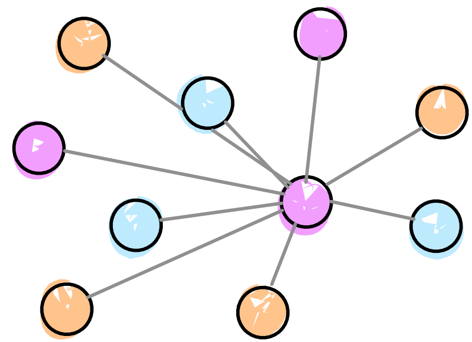

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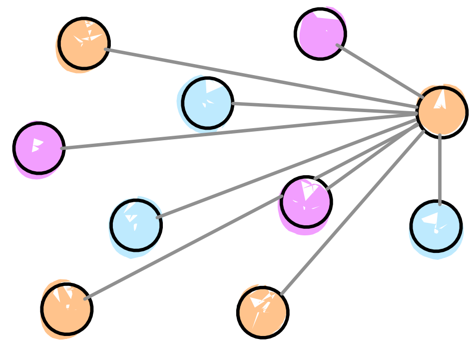

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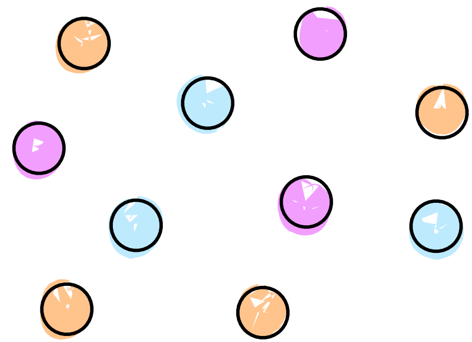


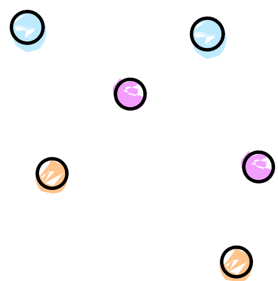
An agent n with strategy u_i has fitness

$$f_i = (1-w)\cdot 1 + w\underbrace{\pi_i}_{\text{expected payoff}} \qquad w \in [0,1]$$

AIM

Limit for:

- Large population: $N \longrightarrow +\infty$
- Small time step: $\tau = t_{h+1} - t_h \longrightarrow 0$
- Weak selection: $w \longrightarrow 0$



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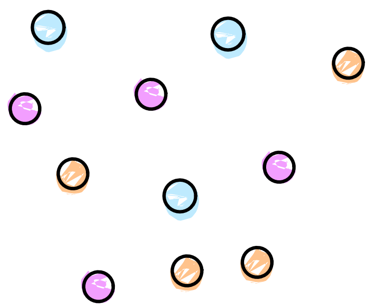

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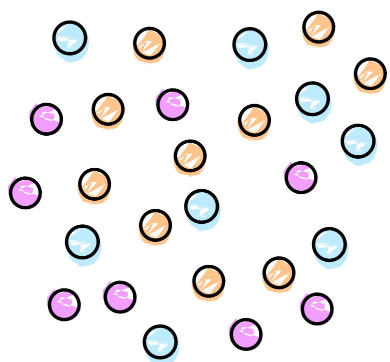


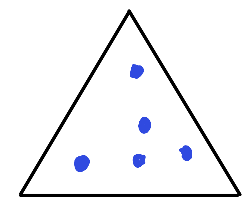
Sequences:

- $\tau_k \longrightarrow 0$
- $N_k = \tau_k^{-\alpha} \longrightarrow +\infty$
- $w_k = \tau_k^{\beta} \longrightarrow 0$

EULERIAN SPECIFICATION

INTERPOLATION:

$$\lambda^k(t) = \lambda^k(t_h) + \frac{t - t_h}{\tau_k}\left(\lambda^k(t_{h+1}) - \lambda^k(t_h)\right)$$



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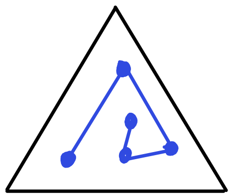

EULERIAN SPECIFICATION

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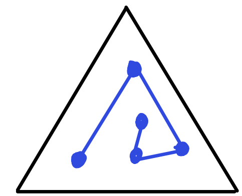
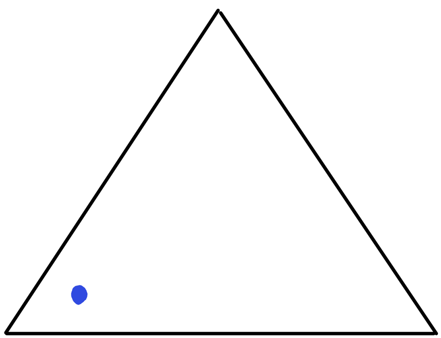


RANDOM PATH:

$$\lambda^{\kappa} : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow C([0,T]; \Delta^{M-1})$$



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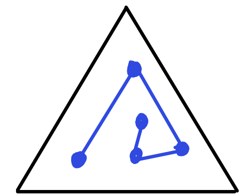

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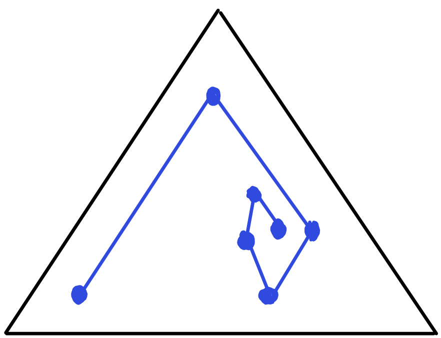

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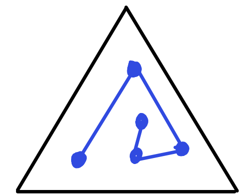

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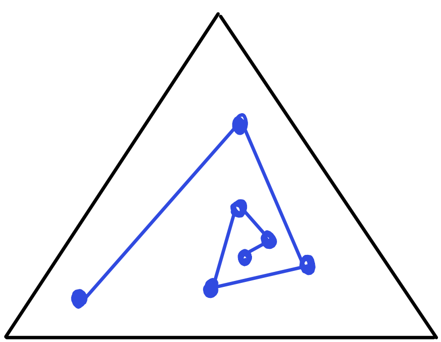

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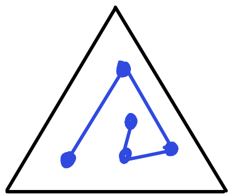

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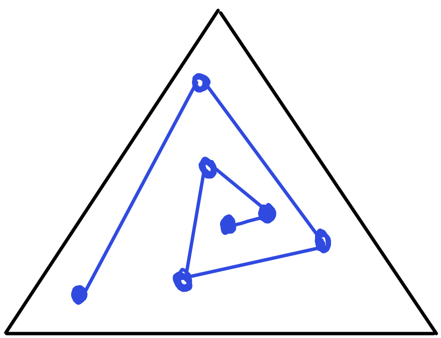

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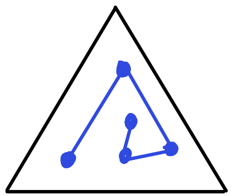

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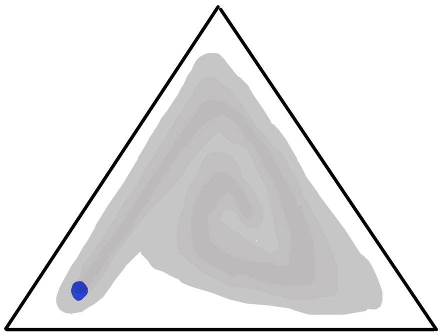


LAW:

$$\Lambda^k = \lambda^k_{\#}\mathbb{P} \in \mathcal{P}\left(C([0,T]; \Delta^{M-1})\right)$$

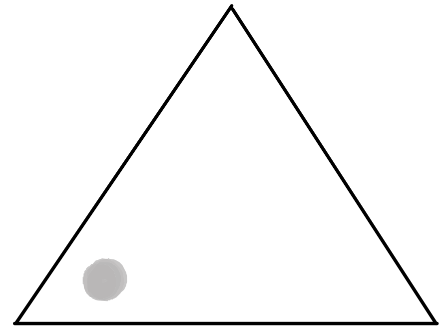
$$\Lambda^k_t = \lambda^k(t)_{\#}\mathbb{P} \in \mathcal{P}(\Delta^{M-1})$$

evolution of Λ^k_t ?

DISCRETE PDE

$$\boxed{\partial_t \Lambda_t^K + \mathrm{div}(b \Lambda_t^K) \approx 0}$$

where $b_i(\lambda) = (A\lambda)_i - \lambda^T A \lambda$



THEOREM: For every test function $\phi \in C_c^\infty([0,T) \times \Delta^{M-1})$

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t,\lambda)\, d\Lambda_t^K(\lambda)\, dt + \frac{w_K}{\tau_K N_K} \int_0^T \int_{\Delta^{M-1}} D\phi(t,\lambda)\, b(\lambda)\, d\Lambda_t^K(\lambda)\, dt =$$

$$= \underbrace{- \int_{\Delta^{M-1}} \phi(0,\lambda)\, d\Lambda_0^K(\lambda)}_{\text{initial datum term}} + \underbrace{\text{“small” error}}_{\text{depending on } \tau_K,\, N_K,\, w_K\,!}$$

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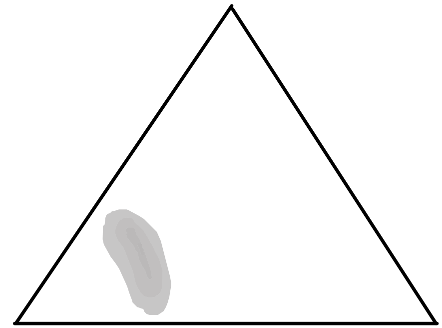

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$$= \underbrace{-\int_{\Delta^{M-1}}\phi(0,\lambda)\,d\Lambda^K_0(\lambda)}_{\text{initial datum term}} + \underbrace{\text{“small” error}}_{\text{depending on } \tau_K,\, N_K,\, w_K\,!}$$

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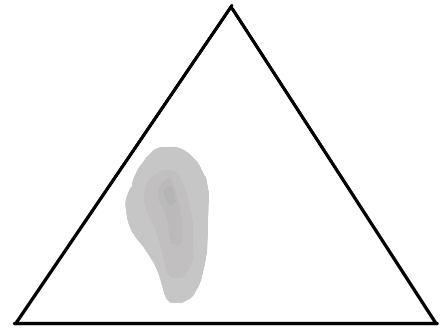

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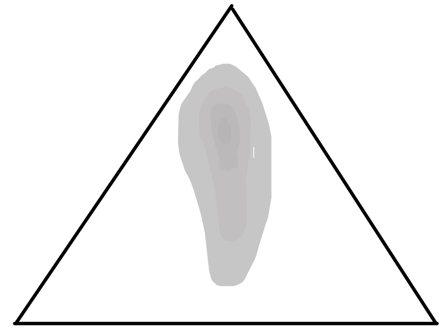
DISCRETE PDE

$$\boxed{\partial_t \Lambda^K_t + \mathrm{div}(b \Lambda^K_t) \approx 0}$$

where $b_i(\lambda) = (A\lambda)_i - \lambda^T A \lambda$



THEOREM: For every test function $\phi \in C^\infty_c([0,T) \times \Delta^{M-1})$

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t,\lambda)\, d\Lambda^K_t(\lambda)\, dt + \frac{w_K}{\tau_K N_K} \int_0^T \int_{\Delta^{M-1}} D\phi(t,\lambda)\, b(\lambda)\, d\Lambda^K_t(\lambda)\, dt =$$

$$= \underbrace{- \int_{\Delta^{M-1}} \phi(0,\lambda)\, d\Lambda^K_0(\lambda)}_{\text{initial datum term}} + \underbrace{\text{“small” error}}_{\text{depending on } \tau_K, N_K, w_K\,!}$$

THE "SMALL" ERROR

Recall the term $\dfrac{w_k}{\tau_k N_k} = \dfrac{\tau_k^{\beta}}{\tau_k \tau_k^{-\alpha}} = \tau_k^{\alpha+\beta-1} \approx 1$

"small" error $= O\left(\dfrac{w_k}{\tau_k N_k^2} \|D\phi\|_\infty\right)$

$+ O\left(\dfrac{w_k^2}{\tau_k N_k} \|D\phi\|_\infty\right)$

$+ O\left(\dfrac{1}{\tau_k N_k^2} \|D^2\phi\|_\infty\right)$!

! $\dfrac{1}{\tau_k N_k^2} = \dfrac{1}{\tau_k \tau_k^{-2\alpha}} = \tau_k^{2\alpha-1}$

ASSUMPTIONS: $\begin{cases} \alpha+\beta = 1 \\ \alpha > \frac{1}{2} \end{cases}$

COMPACTNESS ?

AIM: *Letting* $\tau_k \to 0$, $N_k = \tau_k^{-\alpha} \to +\infty$, $w_k = \tau_k^{\beta} \to 0$ *under the assumptions* $\alpha + \beta = 1$, $\alpha > \frac{1}{2}$ *and passing to the limit*

$$\partial_t \Lambda_t^k + \mathrm{div}(b \Lambda_t^k) \approx 0 \xrightarrow{k \to +\infty} \partial_t \Lambda_t + \mathrm{div}(b \Lambda_t) = 0$$

CAVEAT: *The statement is void without compactness*

REMARK:
- $t \mapsto \Lambda_t^k \in \mathcal{P}(\Delta^{M-1})$
- $(\mathcal{P}(\Delta^{M-1}), W_1)$ *is compact*

IF $t \mapsto \Lambda_t^k$ *equicontinuous*, THEN *use Arzelà-Ascoli.*

COMPACTNESS ?

EQUICONTINUITY?

$\boxed{\alpha+\beta=1,\ \alpha>1/2}$

A rough estimate: $|\lambda^k(t_{h+1})-\lambda^k(t_h)| \leq \frac{\sqrt{2}}{N_k}$ a.s.

$$\Rightarrow \int_{\Delta^{M-1}} \psi(\lambda)\, d(\Lambda^k_{t_{h+1}} - \Lambda^k_{t_h})(\lambda) =$$

$$= \mathbb{E}\left[\psi(\lambda^k(t_{h+1}))-\psi(\lambda^k(t_h))\right] \leq$$

$$\leq \mathbb{E}\left[|\lambda^k(t_{h+1})-\lambda^k(t_h)|\right] \leq \frac{\sqrt{2}}{N_k} = C\tau_k^\alpha$$

$$\Rightarrow \mathcal{W}_1(\Lambda^k_{t_{h+1}}, \Lambda^k_{t_h}) \leq C\tau_k^\alpha$$

COMPACTNESS ?

EQUICONTINUITY?

$\alpha + \beta = 1, \ \alpha > 1/2$

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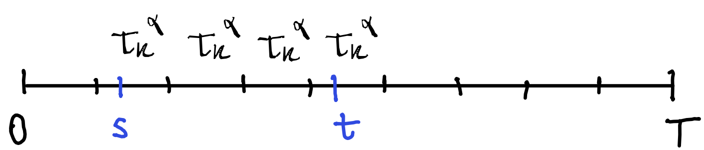
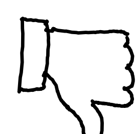
$$\Rightarrow \mathcal{W}_1(\Lambda^k_{t_{h+1}}, \Lambda^k_{t_h}) \leq C\,\tau_k^{\alpha}$$



$$\frac{|t-s|}{\tau_k}\,\tau_k^{\alpha}$$



COMPACTNESS !

IDEA: *Prove it directly.*

- up to a subsequence, $\Lambda_t^k \xrightarrow{\mathcal{W}_1} \Lambda_t$ for every $t \in [0,T] \cap \mathbb{Q}$.
- given $s, t \in [t_h, t_{h+1}] \cap \mathbb{Q}$:

$$\int_{\Delta^{M-1}} \phi(\lambda)\, d(\Lambda_t^k - \Lambda_s^k)(\lambda) \approx$$

$$\approx |t-s| \frac{w_k}{\tau_k N_k} \int_{\Delta^{M-1}} D\phi(\lambda)\, b(\lambda)\, d\Lambda_{t_h}^k(\lambda) +$$

$$+ O\left(\frac{w_k^2}{\tau_k N_k} \|D\phi\|_\infty\right) + O\left(\frac{w_k}{\tau_k N_k^2} \|D\phi\|_\infty\right) + O\left(\frac{1}{\tau_k N_k^2} \|D^2\phi\|_\infty\right)$$

passing to the limit

$$\int_{\Delta^{M-1}} \phi(\lambda)\, d(\Lambda_t - \Lambda_s) \lesssim |t-s| \Rightarrow \mathcal{W}_1(\Lambda_t, \Lambda_s) \lesssim |t-s|.$$

MAIN RESULT

THEOREM: Let $\tau_k \to 0$, $N_k = \tau_k^{-\alpha}$, $w_k = \tau_k^{\beta}$ with $\alpha + \beta = 1$ and $\alpha > 1/2$. Let $\lambda^k : (\Omega, \mathcal{F}, \mathbb{P}) \to C([0,T]; \Delta^{M-1})$ be the random path generated by the discrete Moran process with population of size N_k, payoff matrix A and payoff weight w_k. Let $\Lambda_t^k \in \mathcal{P}(\Delta^{M-1})$ be the law of $\lambda^k(t)$. Then $\Lambda_t^k \xrightarrow{W_1} \Lambda_t$ uniformly in $t \in [0,T]$, where Λ_t is the unique solution to:

$$\partial_t \Lambda_t + \mathrm{div}(b \Lambda_t) = 0 \qquad + \qquad \text{initial datum } \Lambda_0$$

Moreover, $\Lambda_t = \Psi(t,\cdot)_{\#} \Lambda_0$, where $\Psi(t,\cdot)$ is the flow of the vector field b, i.e., $\Psi(t,\lambda_0)$ is the solution to

$$\dot{\lambda}(t) = b(\lambda(t)) \qquad + \qquad \lambda(0) = \lambda_0$$

THANK YOU
FOR THE ATTENTION

ESSENTIAL LITERATURE:

- M. Morandotti, G. Orlando. Preprint 2025.
- M. A. Novak. "Evolutionary dynamics: Exploring the equations of life (2006)
- F. A. C. C. Chalub, M. O. Souza. Teorethical Population Biology (2009)
- L. Ambrosio, M. Fornasier, M. Morandotti, G. Savaré. CPAM (2021)