The role of Artificial Intelligence in the future of mathematics

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CERMICS - Ecole des Ponts Paristech

FAU MoD Lecture

–

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École des Ponts ParisTech

Introduction

Social networks

Emails and apps

Virtual assistants

Platforms

Introduction

Introduction

Over the past year, we have heard a lot about the progress of AI, particularly in one field: AI for language.

Introduction

How did we get there ?

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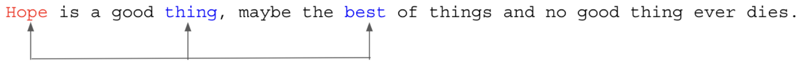
- AI for language is not new: research started ∼1950
- A turning point in the years 2010s with the neural networks (large progress in translation, etc.)
- Another turning point in 2017: the Transformer.

A turning point in 2017: the Transformer

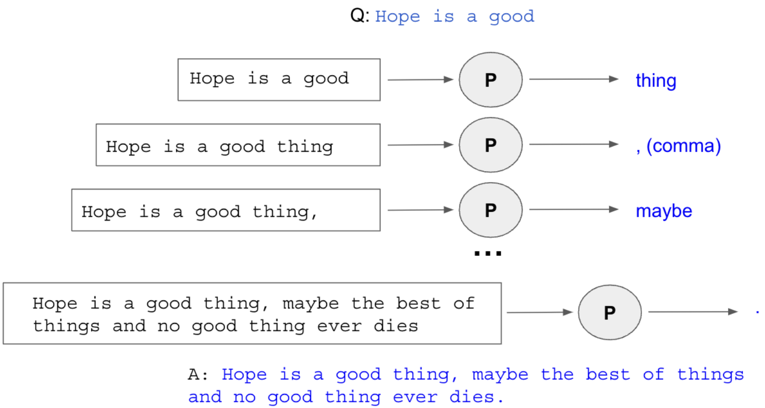
Attention is all you need
A Vaswani, N Shazeer, N Parmar… - Advances in neural …, 2017 - proceedings.neurips.cc
… to attend to **all** positions in the decoder up to and including that position. **We need** to prevent … **We** implement this inside of scaled dot-product **attention** by masking out (setting to $-\infty$) …

- An attention mechanism, which allows it to focus on the right pieces of a sentence

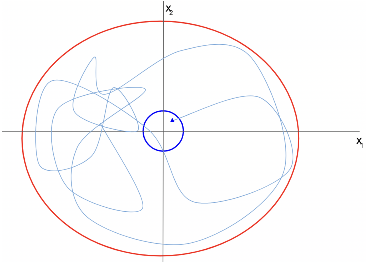


Introduction

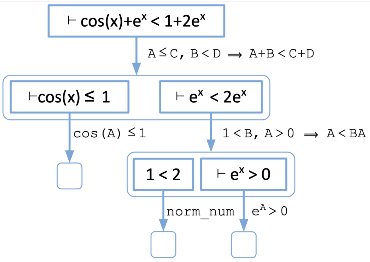
2018 - GPT - an autoregressive transformer.



Outline of the Lecture

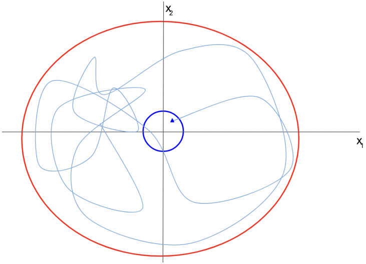


1. AI today in mathematics



2. Can AI prove theorems?

Outline of the Lecture



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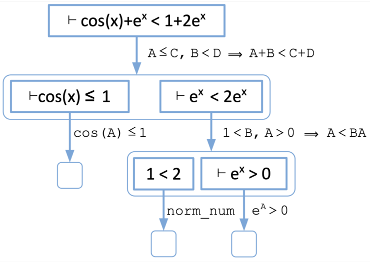

2. Can AI prove theorems?

Solving maths problems with a computer

Solving maths problems with a computer

Conjecture (Euler, 1769)

If there exist integers a_1, a_2,...,a_k, b, and n such that

$$a_1^n + a_2^n + \ldots + a_k^n = b^n,$$

then $k \geq n$.

A problem open for almost 200 years

Solving maths problems with a computer

Lander and Parkin (1966)

Solving maths problems with a computer

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$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Solving maths problems with a computer

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n > 2$.

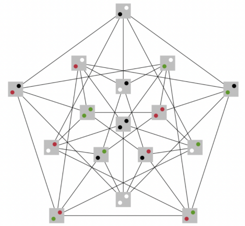
REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

Solving maths problems with a computer

Proof of Keller's Conjecture (Brakensiek, Heule, Mackey, Narváez, 2019)

- A proof with many "simple cases" to check
- Many = far too many for a human



Solving maths problems with a computer

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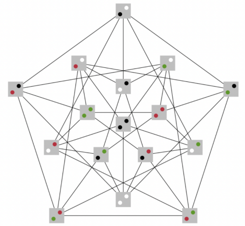

Solving maths problems with a computer

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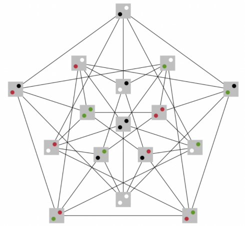

Conclusion: Computers have been used to prove theorems for a long time.

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Can AI solve more complicated problems?

Problems where the difficulty is not just combinatorial?

AI in Mathematics Today

Three examples

- Stability of dynamical systems
- Control theory
- Topology

AI in Mathematics Today

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Stability of Dynamical Systems

A system of differential equations

$$\dot{x}(t) = f(x(t)),$$

Stability of Dynamical Systems

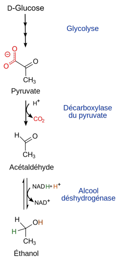
A point in free fall.

$$\dot{y}(t) = v(t)$$
$$\dot{v}(t) = -g,$$

Stability of Dynamical Systems

The evolution of a chemical reaction

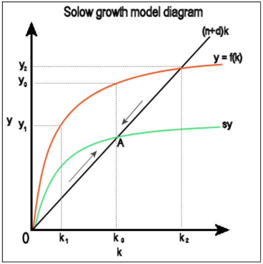
$$\dot{x} = -\alpha x(t) + (1 - x(t)),$$
$$\dot{y} = \alpha x(t) - y(t),$$



Stability of Dynamical Systems

The Solow model in economics

$$\dot{K}(t) = sF(K(t), aL(t)) - \delta K v(t)$$
$$\dot{L}(t) = nL(t),$$



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Question (System Stability)

Is it true that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial condition satisfies $\|x(0)\| \leq \delta$ then the solution $x(t)$ exists for all $t \in [0, +\infty)$ and

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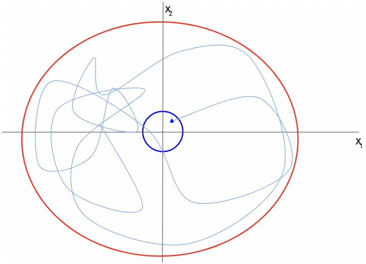
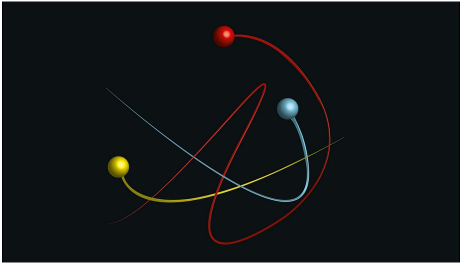


Stability of Dynamical Systems

A problem that has interested mathematicians for over a hundred years.



Stability of Dynamical Systems

A significant advancement: Lyapunov functions

Theorem

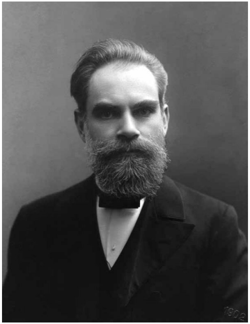
If there exists a function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that for all* $x \in \mathbb{R}^n$

$$V(x) > V(0), \quad \textit{and} \quad \nabla V(x) \cdot f(x) \leq 0,$$

and

$$\lim_{\|x\| \to +\infty} V(x) = +\infty,$$

then the system is stable.



A. Lyapunov (1857-1918)

Stability of Dynamical Systems

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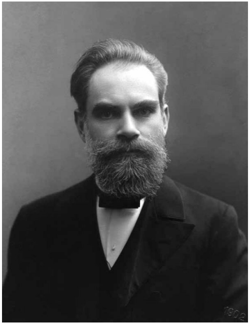

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Stability of Dynamical Systems

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$$\dot{x}(t) = \begin{pmatrix} -6x_1^4(t)x_2^5(t) - 3x_1^7(t)x_3^2(t) \\ 3x_1^9(t) - 6x_1^2(t)x_2^5(t)x_3^2(t) \\ -4x_1^2(t)x_3^5(t) \end{pmatrix}$$

Stability of Dynamical Systems

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The system is stable, a Lyapunov function is

$$V(x) = x_1^6 + 2(x_2^6 + x_3^4)$$

Stability of Dynamical Systems

A globally asymptotically stable polynomial vector field with no polynomial Lyapunov function

Publisher: IEEE

Cite This

PDF

Amir Ali Ahmadi ; Miroslav Krstic ; Pablo A. Parrilo All Authors

Stability of Dynamical Systems

Today, more than a hundred years later, it is still an open question:

there is still no systematic way to construct a Lyapunov function.

Stability of Dynamical Systems

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$\longrightarrow$ We resort to intuition

Stability of Dynamical Systems

- Intuition, an important concept in mathematics.

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- In a number of cases, intuition resembles a kind of pattern recognition. You've seen plenty of examples, and this gives you an idea of how to proceed in a case you've never seen.

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First task for AI: guessing Lyapunov functions

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First task for AI: guessing Lyapunov functions

$$\dot{x}(t) = \begin{pmatrix} -6x_1^4(t)x_2^5(t) - 3x_1^7(t)x_3^2(t) \\ 3x_1^9(t) - 6x_1^2(t)x_2^5(t)x_3^2(t) \\ -4x_1^2(t)x_3^5(t) \end{pmatrix} \quad \rightarrow \quad \text{Yes, } V(x) = x_1^6 + 2(x_2^6 + x_3^4)$$

Stability of Dynamical Systems

Train an AI to have an intution of Lyapunov function (Alfarano, Charton, A.H., 2023).

Neural network architecture: Transformer (∼1000 smaller than ChatGPT)

Procedure:

1. Generate a set of systems and associated Lyapunov functions.
2. Encode the examples
3. Train the Transformer (supervised learning)

Stability of Dynamical Systems

Procedure:

1. Generate a set of systems and associated Lyapunov functions.
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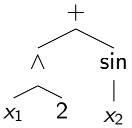
$(x_1^2+\sin(x_2)) \quad \rightarrow$



$\rightarrow \quad "+", "\wedge", "x_1", "2", "\sin", "x_2"$

3. Train the Transformer (supervised learning)

Stability of Dynamical Systems

Results

Type	n equations	degree	SOSTOOLS[1]	AI
polynomial	2-3	8	78%	**99.3%**
polynomial	3-6	12	16%	**95.1%**
polynomial (fwd)	2-3	6	N/A	**83.1%**
Non-polynomial	N/A	N/A	N/A	**97.8%**

[1] Existing method.

Stability of Dynamical Systems

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Mathematicians accuracy: $\sim$ 25%

[1] Existing method.

AI in Mathematics Today

Three examples

- Stability of dynamical systems
- Control theory
- Topology

Control of a Differential System

Evolution of the mosquito population

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E \frac{M}{M + M_s} - \delta_F F, \\ \dot{M}_s = u - \delta_s M_s, \end{cases}$$

$E(t)$ represents mosquito eggs, $F(t)$ fertilized females, $M(t)$ males, $M_s(t)$ sterile males. u is the flow of sterile mosquitoes that we release. This is what is called *control*.

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$$u = f(M + M_s, F + F_s)$$

with $F_s = FM/M_s$.

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Question

Is it possible to find f such that the system is stable and

$$\lim_{t \to +\infty} \|E(t), M(t), F(t)\| = 0 \text{ and } \lim_{t \to +\infty} \|M_s(t)\| = \varepsilon,$$

with ε as small as desired?

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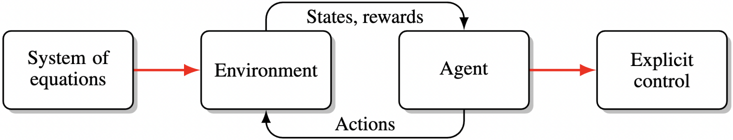
An open question

Control of a Differential System

Principle of the approach (Agbo Bidi, Coron, A.H., Lichtlé, 2023)



Control of a Differential System

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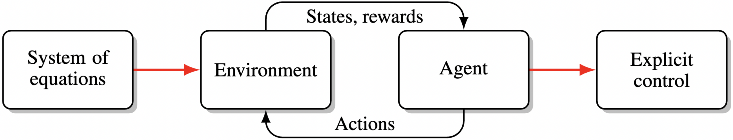
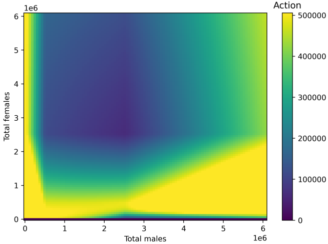


1. **Transform the equations** using a well-chosen numerical scheme.
2. **Train a Reinforcement Learning (RL) model**. The AI trains by trial and error and tries to maximize a well-chosen objective.
3. **Deduce the mathematical control**, from the numerical control.
4. **Verify** that it is a solution to the problem.

Control of a Differential System



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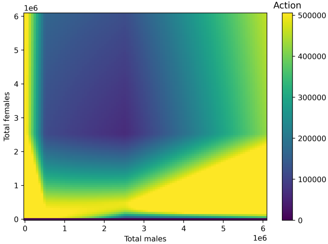
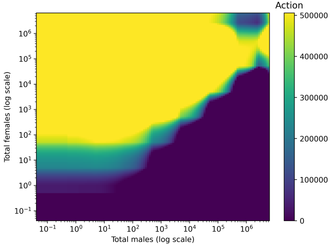
Control of a Differential System



$\xrightarrow{\text{N.Lichtlé}}$



$$u = f(M + M_s, F + F_s)$$

Control of a Differential System

$$u_{\text{reg}}(M+M_s,F+F_s)=\begin{cases}u_{\text{reg}}^{\text{left}}(M+M_s,F+F_s) & \text{if } M+M_s<M^*,\\ u_{\text{reg}}^{\text{right}}(M+M_s,F+F_s) & \text{otherwise,}\end{cases}$$

Control of a Differential System

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$$u_{\text{reg}}^{\text{left}} = \begin{cases} \varepsilon & \text{if } l_1(F+F_s) > \alpha_2, \\ u_{\max}\,(\alpha_2 - l_1) & \text{if } l_1 \in (\alpha_1, \alpha_2], \\ u_{\max} & \text{otherwise, and} \end{cases}$$

$$u_{\text{reg}}^{\text{right}} = \begin{cases} \varepsilon & \text{if } l_2 > \alpha_2, \\ u_{\max}\,(\alpha_2 - l_2) & \text{if } l_2 \in (\alpha_1, \alpha_2], \\ u_{\max} & \text{otherwise.} \end{cases}$$

where $l_1(x) = \frac{\log M^*}{\log(F+F_s)}$ and $l_2(x,y) = \frac{\log(M+M_s)}{\log(F+F_s)}$,

Control of a Differential System

Final control

$$u(t) = \begin{cases} \varepsilon & \text{if } \frac{\log(M+M_s)}{\log(F+F_s)} > \alpha_2, \\ u_{\max} & \text{otherwise,} \end{cases}$$

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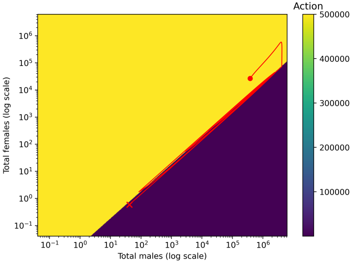

$$\varepsilon > 0$$

Control of a Differential System

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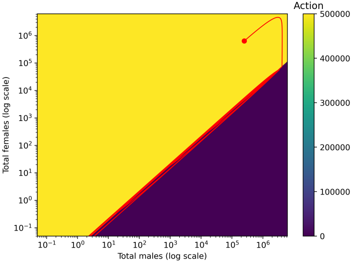

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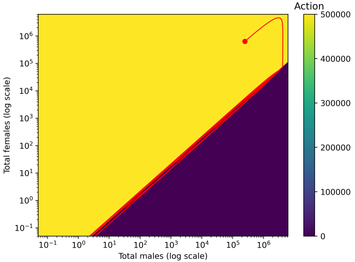

$\varepsilon = 0$

We can see a mathematical bifurcation with our "AI-augmented intuition".

AI Today in Mathematics

Three examples

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Topology

Advancing mathematics by guiding human intuition with AI, 2021, Davies et al.

Principle:

- We have mathematical objects z with quantities $X(z)$ and $Y(z)$. We would like to know if there is a link between the two.

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Principle:

- We have mathematical objects z with quantities $X(z)$ and $Y(z)$. We would like to know if there is a link between the two.
- We train a neural network to predict $Y(z)$ from $X(z)$
- We try to understand the function learned by the neural network

$$Y(z) = \hat{f}(X(z)).$$

Topology

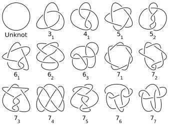
Advancing mathematics by guiding human intuition with AI, 2021, Davies et al.

Principle:

- We have mathematical objects z with quantities $X(z)$ and $Y(z)$. We would like to know if there is a link between the two.
- We train a neural network to predict $Y(z)$ from $X(z)$
- We try to understand the function learned by the neural network

$$Y(z) = \hat{f}(X(z)).$$

In particular, the link between hyperbolic and algebraic invariants of knots (embedding of a circle in $\mathbb{R}^3$).



AI in Mathematics Today

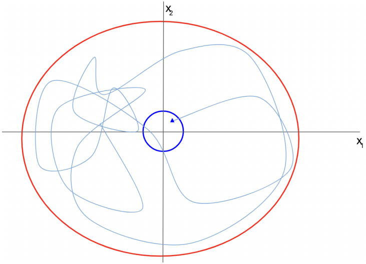
- AI is already useful in the practice of mathematics and has solved several difficult problems.
- AI is trained to have better intuition than humans on a specific problem.
- This augmented intuition allows us to bypass the difficulty of the problem.

Future of Mathematical AI

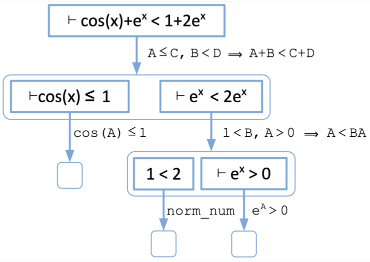
Can an AI prove a mathematical result on its own?

Can an AI reason?

Outline of the Presentation



1. AI today in mathematics



2. Can AI prove theorems?

AI for Mathematical Proof

- Can an AI find a proof for a mathematical statement?

- Related question: can we automate human reasoning?

AI for Mathematical Proof

- Can an AI find a proof for a mathematical statement? Many research groups around the world (Ecole des Ponts, Cambridge, Meta AI, OpenAI, etc.)
- Related question: can we automate human reasoning? A research group led by Timothy Gowers in Cambridge.

AI for Mathematical Proof

First approach: training a Transformer (GPT-f, Polu, Sutskever, 2020)

Question

Let $a > 0$ *and* $b > 0$*, such that* $ab = b - a$*, show that*

$$\frac{a}{b} + \frac{b}{a} - ab = 2$$

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Ilya Sutskever: The OpenAI Genius Who Told Sam Altman He Was Fired

Company's chief scientist led a board coup against one of the most prominent figures in Silicon Valley

Ilya Sutskever SUIVRE

Co-Founder and Chief Scientist of OpenAI
Adresse e-mail validée de openai.com - Page d'accueil
Machine Learning Neural Networks Artificial Intelligence Deep Learning

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Citations	462122	356473
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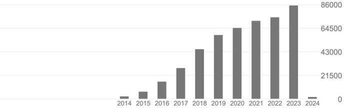

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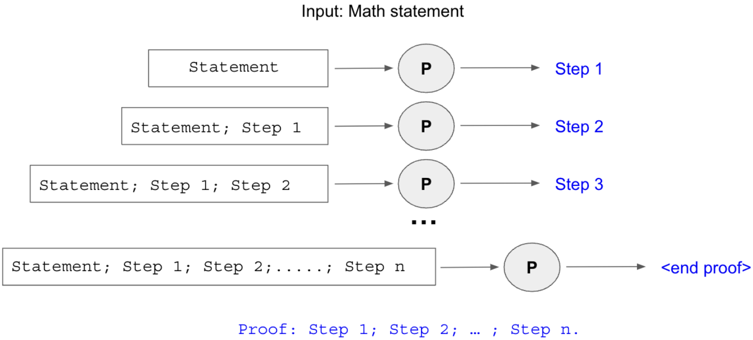

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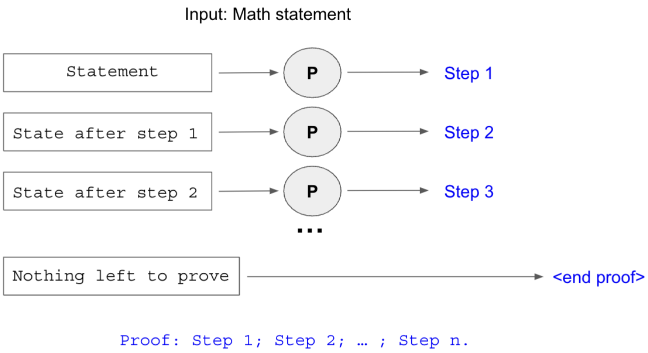

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Enough = sufficiently diverse and sufficiently numerous

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We have very few data available (especially formal).

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informal language

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sorry,
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LeanLlama F. Glöckle et al. 2023 (Temperature-scaled large language models for Lean proofstep prediction)

AI for Mathematical Proof

Second approach: treat mathematics as a game (Lample, Lachaux, Lavril, Martinet, Hayat, Ebner, Rodriguez, Lacroix, 2022)

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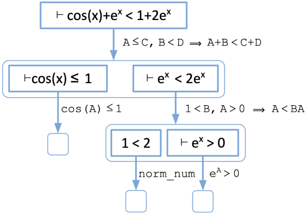
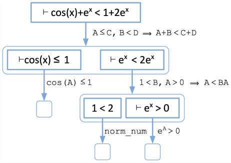


You won !

AI for Mathematical Proof

Main difficulties:

- two-player game vs. solo against a goal.
- In chess, when you play a move you always have a single game. In mathematics: one statement $\rightarrow$ multiple statements
- Difficult in mathematics to know automatically in the middle of a proof what the probability of succeeding is.
- The number of possibilities is much, much larger in mathematics



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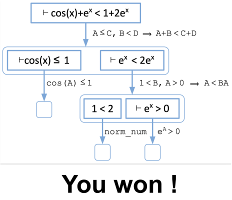


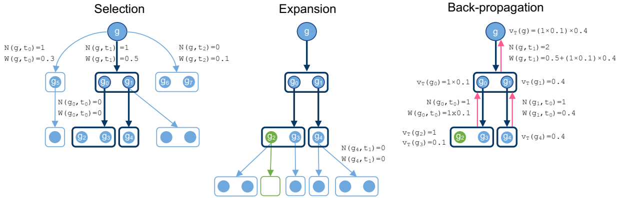
Much more difficult than chess

AI for Mathematical Proof

In practice

- Two transformers: P_θ which predicts a tactic, c_θ which predicts the difficulty of proving a statement (goal, hypothesis, etc.).

- An intelligent proof search that sees the proof as a tree and combines P_θ, c_θ and a tree expansion.



- Training of P_θ and c_θ as and when what has been successful

AI for Mathematical Proof

Results

Exercises at the undergraduate level...

...30 to 60% of middle school / high school exercises up to Olympiad level...

...and a few exercises from the International Mathematical Olympiads

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Unexpected outcomes

French AI start-up Mistral reaches unicorn status, marking its place as Europe's rival to OpenAI

Mistral's value has increased more than sevenfold in six months. It raised nearly €500 million in November and €105 million in its first funding round.

(Euronews 11/12/2023)

AI for Mathematical Proof

Perspectives:

- Add (approximations of) human reasoning.
- Add more organized reasoning with a hierarchy of tasks and a foresight of intermediate tasks.
- Obtain more formal data through self-formalization (Wu et al., Jiang et al. 2022).

Conclusion

- AI is already useful in the practice of mathematics
- AI for proving theorems is only beginning, and there are many ideas... and much to do.
- The practice of mathematics will probably change... and that's okay.
- AI will not replace mathematicians but will instead enhance them.

Conclusion

Thank you for your attention