Analysis of commuting times in a linear city

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based on ongoing project with
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The phenomenon of daily commuting home-work

Fujita Okawa model urban model ('80)- and subsequent agent based models. The spatial configuration of the city is the outcome of interactions of two large populations of agents, workers and firms (competing for space/ interacting through labour market).

- Optimal transport approach:
  competition for the land (relative density of residents depend on wage-commuting cost whereas level of employment depend on production -wage): revenue ϕ and wage ψ are conjugate through the commuting cost

$$\begin{cases} x \in \mathrm{supp}\mu_I, & \phi(x) = \sup\{\psi(y) - c(x,y) \quad y \in \mathrm{supp}\mu_F\} \\ y \in \mathrm{supp}\mu_F, & \psi(y) = \sup\{\phi(x) - c(x,y) \quad x \in \mathrm{supp}\mu_I\} \end{cases}$$

  $+$ penalization of concentrations of people
  Optimal transportation/ Nash equilibria **Buttazzo, Santambrogio 2005, Carlier, Santambrogio 2005, Crippa, Jimenez, Pratelli, 2009, Carlier Ekeland 2009**

- Mean-field game approach: agents of each population solve a control problem where the cost functional depends on the distribution of similar agents and the interaction with the ones of the other population (rent - revenue/rent+wage), **Barilla, G. Carlier, J.-M. Lasry, 2021, Achdou, Carlier, Petit, Tonon, 2023, Camilli, Festa, Marzufero 2025**

Our approach

The location choice and the rent cost are strongly related to commutation cost, including the actual fuel+vehicle expenses, as well as indirect costs, such as the time of traveling (strongly impacted by congestion), the discomfort, and so on: What is the "cost" of commuting?

- We look at a daily time scale: No competition for land between firms and households that is the distribution of workers μ_I and the distribution of firms to be reached by the workers μ_F along with living place-working place assignement are given.
- We simplify the model passing from a bidimensional setting (with radial symmetry or without), to 1-dimensional (linear) city model (first step towards a network setting).
- Fluid dynamic approach to model the phenomenon of commuting with congestion, where the distribution of workers moving from home to work is evolving according to a balance law.

Our aim: Provide a mathematical model that describes the daily phenomenon of commutations taking into account the congestion effects
More generally: develop a theoretical setting to deal with either micro or macro multiagent models with incoming and outgoing fluxes

LWR model for traffic

Classical Lighthill-Whitham-Richards model:
$\bar{\rho} \in L^\infty(\mathbb{R}, [0, +\infty))$ with compact support in $[0, L]$, $\|\bar{\rho}\|_\infty \leq M$, $\|\bar{\rho}\|_1 = 1$ is the initial distribution of cars,
which evolves according to the CL

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 & \text{in } (0, T) \times \mathbb{R}, \\ \rho(0, x) = \bar{\rho}(x) & \text{in } \mathbb{R}. \end{cases}$$

where the flux is " density times average speed", with the speed which satisfies

$$v : [0, M] \to [0, v_{\max}] \qquad C^1 \text{ and strictly decreasing,: } v(0) = v_{\max}, v(M) = 0$$

also $\rho \to \rho v'(\rho)$ non increasing.
Typical examples:

$$v(\rho) = v_{\max}\left(1 - \frac{\rho^\alpha}{M^\alpha}\right), \qquad v(\rho) = \frac{v_{\max}}{\log(M+1)} \log\left(\frac{M+1}{\rho+1}\right).$$

Balance law with measure sources with compatibility conditions

$$\begin{cases} \mu_I\text{- distribution of workers living places} \\ \mu_F\text{- distribution of firms} \end{cases}$$

probability measures supported in $[0, L]$

We introduce the balance law + compatibility conditions

$$\begin{cases} (BL) & \partial_t \rho + \partial_x(\rho v(\rho)) = m_I - m_F, \\ (CC) & (\pi_x)_\sharp m_I = \mu_I, \quad (\pi_x)_\sharp m_F = \mu_F. \end{cases}$$

Given μ_I, μ_F, the unknown is the triple (ρ, m_I, m_F) s.t.

- m_I, m_F are positive Radon measures supported on $[0, L] \times [0, T]$
  - $m_I([x_1, x_2] \times [t_1, t_2])$ is the "amount of drivers" that enter in the road at some time $t \in [t_1, t_2]$ somewhere in the interval $[x_1, x_2]$,
  - $m_F([x_1, x_2] \times [t_1, t_2])$ is the "amount of drivers" that exit in the road at some time $t \in [t_1, t_2]$ somewhere in the interval $[x_1, x_2]$,
- $\rho \in L^\infty$, supported in $[0, T] \times [0, L]$ with $\|\rho\|_\infty \leq M$ is the density of drivers on the road, is a (m_I, m_F)-entropy solution

Balance law with measure sources

$$(BL) \qquad \partial_t \rho + \partial_x(\rho v(\rho)) = m_I - m_F$$

Definition (Balance law with measure valued sources)

Given two finite positive measures $m_I, m_F \in \mathcal{P}(\mathbb{R}^2)$ supported in $[0, T] \times \mathbb{R}$ we say that $\rho \in L^\infty([0, T] \times \mathbb{R})$ is a (m_I, m_F)-entropy solution if

- ρ is a solution of (BL) in the sense of distribution
- ρ satisfies in the sense of distributions of the entropy inequality:

$$(E) \qquad \partial_t|\rho - k| + \partial_x(\operatorname{sign}(\rho - k)(f(\rho) - f(k)) \leq m_I + m_F \qquad \forall k.$$

Many properties

- more generally for weakly genuinely nonlinear fluxes
- no uniqueness
- Let ρ_n be a $(m_{I,n}, m_{F,n})$-entropy sln for every $n \in \mathbb{N}$ and assume that $\rho_n, m_{I,n}, m_{F,n} \rightharpoonup^* \rho, m_I, m_F$. Then ρ is a (m_I, m_F)-entropy sln.
- Let $L^\infty(\mathbb{R}^2) \times \left(\mathcal{P}(\mathbb{R}^2)\right)^2$ with the product of the L^1_{loc} and the weak* topology. The subsets of triples: (ρ, m_I, m_F) with m_I, m_F supported on $[0, T] \times [0, L]$, ρ a (m_I, m_F)-entropy sln supported on $[0, T] \times [0, L]$ with $\|\rho\|_{L^\infty} \leq M$ is compact.
  consequence of Kruzkov-type estimate, and of the (fractional)BV-like regularity properties of entropic solutions (**Marconi, (J. Hyp Diff. Eq. 2018)**).

Lagrangian formulation

Observation: For the application to the analysis of commuting times, we are interested in solutions which are compatible not only with

$$\begin{cases} \mu_I\text{- distribution of workers living places} \\ \mu_F\text{- distribution of working places} \end{cases}$$

but also with

$$\text{a plan } \pi \in \mathcal{P}(\mathbb{R}^2) \text{ with } \pi(A \times \mathbb{R}) = \mu_I(A), \pi(\mathbb{R} \times B) = \mu_F(B).$$

The plan takes into account the living place-working place assignement

We develop a Lagrangian formulation based on microscopic model (the celebrated Follow the Leader)

Lagrangian description of LWR model/ particle path formulation

Let

$$\Gamma = \{\gamma \in C^{0,1}([0,T];\mathbb{R})\ 0 \le \dot\gamma \le v_{\max} \text{ a.e.}\} \qquad \text{with topology unif. conv.}$$

$\mathcal{M}(\Gamma)$ positive Radon measures on Γ with the topology of weak* convergence.
Let $e_t : \Gamma \to \mathbb{R}$, for $t \in [0,T]$ evaluation map, that is $e_t(\gamma) := \gamma(t)$
Among curves in Γ, we are interested in the characteric ones:

$$\Gamma_{char} := \{\gamma \in \Gamma \text{ with } \dot\gamma(t) = v(\rho(t,\gamma(t))) \text{ for a.e. } t \in [0,T]\}.$$

Theorem

Let ρ the unique entropy solution to the (CL) with i.d. $\bar\rho$.
$\exists\ \omega \in \mathcal{M}(\Gamma)$ *and* $X(t,y) : [0,T] \times \mathbb{R} \to [0,L]$ *meas. and Lipschitz in t, s.t.*

- *for $\bar\rho\mathcal{L}^1 -$ a.e. $y \in [0,L]$ it holds $X(0,y) = y$ and*

$$\partial_t X(t,y) = v(\rho(t, X(t,y)^-)) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,T).$$

- $e_t\sharp\omega = \rho(t,\cdot)\mathcal{L}^1 = X(t,\cdot)_\sharp\bar\rho\mathcal{L}^1$
  that is for all t, y, $\int_{-\infty}^{y} \bar\rho(x)dx = \int_{-\infty}^{X(t,y)} \rho(t,x)dx = \omega(\gamma, \gamma(t) \le X(t,y))$.

So ω is concentrated on $\{X(\cdot,y) : [0,T] \to [0,L], y \in [0,L]\} \subseteq \Gamma_{char}$.

Ref: Fjordholm, Mæhlen, Ørke, (2024)

Lagrangian viewpoint: commuting strategies compatible with a plan

We adopt a Lagrangian viewpoint, describing the commuting strategies as measures on Lipschitz curves each associated with an entry time on the road and an exit time from the road, which are compatible with the given initial plan among the distribution of lodging and workplace locations, respectively.

Fix a plan $\pi \in \mathcal{P}(\mathbb{R}^2)$ plan between μ_I, μ_F, taking into account the living place-working place assignement, that is: $\pi(A \times \mathbb{R}) = \mu_I(A)$, $\pi(\mathbb{R} \times A) = \mu_F(A)$.

The set Γ of curves becomes:

$$\Gamma = \{(t_{\text{in}}, t_{\text{ou}}, \gamma),\ \gamma \in C^{0,1}([t_{\text{in}}, t_{\text{ou}}]; \mathbb{R})\ 0 \leq \dot{\gamma} \leq v_{\max} \text{ a.e.}, -\infty < t_{\text{in}} \leq t_{\text{ou}} < +\infty\}$$

As before $\mathcal{M}(\Gamma)$ the space of positive Radon measures on Γ, endowed with the topology associated to the weak star convergence of measures.

The evaluation maps:

$$\begin{cases} e_t : \Gamma \to \mathbb{R} & e_t(t_{\text{in}}, t_{\text{ou}}, \gamma) = \gamma(t) \\ e_p : \Gamma \to \mathbb{R}^2 & e_p(t_{\text{in}}, t_{\text{ou}}, \gamma) = (\gamma(t_{\text{in}}), \gamma(t_{\text{ou}})) \\ e_{\text{in}} : \Gamma \to \mathbb{R}^2 & e_{\text{in}}(t_{\text{in}}, t_{\text{ou}}, \gamma) = (t_{\text{in}}, \gamma(t_{\text{in}})) \\ e_{\text{ou}} : \Gamma \to \mathbb{R}^2 & e_{\text{ou}}(t_{\text{in}}, t_{\text{ou}}, \gamma) = (t_{\text{ou}}, \gamma(t_{\text{ou}})). \end{cases}$$

Admissible commuting strategies $\mathcal{K}_{L,M,T}(\pi)$

Let $(L, M, T) \in (0, +\infty)^3$ fixed. Let $\omega \in \mathcal{M}(\Gamma)$ and define

$$m_I = e_{\mathrm{in}}\sharp\omega, \qquad m_F = e_{\mathrm{ou}}\sharp\omega.$$

We say that ω is a (L, M, T)-admissible commuting strategy if the following holds:

1. the measure $e_t\sharp\omega\llcorner\Gamma_t$ where $\Gamma_t = \{(t_{\mathrm{in}}, t_{\mathrm{ou}}, \gamma) \in \Gamma : t \in [t_{\mathrm{in}}, t_{\mathrm{ou}}]\}$ is absolutely continuous with respect to Lebesgue and has density $\rho(t, \cdot)$
2. (ρ, m_I, m_F) is a solution to $(BL) + (CC)$
3. ω is concentrated on the set

   $$\Gamma_{\mathrm{char}} := \{(t_{\mathrm{in}}, t_{\mathrm{ou}}, \gamma) \in \Gamma \text{ with } \dot{\gamma}(t) = v(\rho(t, \gamma(t))) \text{ for a.e. } t \in [t_{\mathrm{in}}, t_{\mathrm{ou}}]\}$$

   that is $\omega(\Gamma \setminus \Gamma_{\mathrm{char}}) = 0$.
4. $e_p\sharp\omega = \pi$.

Properties of $\mathcal{K}_{L,M,T}(\pi)$: compactness

Theorem

If $\omega_n \in \mathcal{K}_{L,M,T}(\pi)$ are admissible commuting strategies such that $\omega_n \rightharpoonup^* \omega$ then $\omega \in \mathcal{K}_{L,M,T}(\pi)$.
In particular $\mathcal{K}_{L,M,T}(\pi)$ is compact in $\mathcal{M}(\Gamma)$ endowed with the weak star topology of measures.

Idea of proof:

1. $m_{n,I} \rightharpoonup^* e_{\mathrm{in}}\sharp\omega := m_I$, $m_{n,F} \rightharpoonup^* e_{\mathrm{ou}}\sharp\omega := m_F$ and $e_{t\sharp}\omega_n \llcorner \Gamma_t \rightharpoonup^* e_{t\sharp}\omega \llcorner \Gamma_t$ since $e_{t\sharp}\omega_n \llcorner \Gamma_t = \rho_n(t,\cdot)\mathcal{L}^1$, and ρ_n are equibounded in L^∞, it holds $\rho_n(t,\cdot) \rightharpoonup^* \rho(t,\cdot)$ and $e_{t\sharp}\omega \llcorner \Gamma_t = \rho(t,\cdot)\mathcal{L}^1$.
   By closure by weak conv. (ρ, m_I, m_F) is a $(BL)+(CC)$ solution.
2. the sequence (ρ_n) is also compact in L^1 by the compactness of triples (ρ, m_I, m_F), with ρ a (m_I, m_F)-entropy sln, m_I, m_F supported in $[0,T]\times[0,L]$. Therefore $\rho_n \to \rho$ in $L^1([0,T]\times[0,L])$.
3. by L^1 convergence of $\rho_n \to \rho$ and a Fubini-Tonelli type argument, if ω_n is concentrated in Γ_{char} also ω is concentrated there.

Properties of $\mathcal{K}_{L,M,T}(\pi)$: it is not empty

Theorem

Let π a plan between μ_I, μ_F, supported in $[0, L]^2$ which satisfies the compatibility condition:

$$\textit{if } (x, y) \in \textit{supp } \pi \textit{ then } x \leq y.$$

Assume

$$T > T_m = \min_{c \in (0,M)} \frac{1}{cv(c)} + \frac{L}{v(c)}.$$

Then there exists at least one admissible commuting strategy $\omega \in \mathcal{K}_{L,M,T}(\pi)$.

1. first compatibility condition due to the fact that in our model agents move with positive speed
2. second compatibility condition due to finite speed of propagation. The value T_m takes into account the maximal distance L that drivers may travel and the flux $\rho v(\rho)$ of vehicles on the road.
3. Proof based on compactness and on appropriate modification of the FtL microscopic model with entry and exit.

Commutation costs

Typical commutation costs:

- the L^p norm of the total time:

$$T_p(\omega)^p = \int_\Gamma |t_{ou} - t_{in}|^p \, d\omega(t_{ou}, t_{in}, \gamma)$$

$$T_\infty(\omega) = \sup\{|t_{ou} - t_{in}|, (t_{ou}, t_{in}, \gamma) \in \mathrm{supp}\ \omega\}.$$

- the average travel time from x_1 to x_2 for a given commuting strategy:

$$\tau_{x_1,x_2}(\omega) = \frac{1}{T}\int_0^T \int_{x_1}^{x_2} \frac{1}{v(\rho(t,x))} dxdt$$

  where $\rho(t,\cdot)\mathcal{L}^1 = e_t \sharp \omega$.

- Other possible costs can be considered, where the functional to be optimized by the drivers takes into account a cost for starting early, a cost for arriving late and the total time spent in the car.

By non-emptiness and compactness of the set of admissible strategies, and by lsc of the functionals, there exists always an optimal strategy minimizing each of the previous functionals

Open problems

1. minimal conditions for uniqueness of the commuting strategies: we conjecture that if $\omega_1, \omega_2 \in \mathcal{K}_{L,M,T}(\pi)$ satisfy $e_{\mathrm{in}}\sharp\omega_1 = m_I = e_{\mathrm{in}}\sharp\omega_2$, then $\omega_1 = \omega_2$.
2. Qualitative description of optimal strategies
3. Develop the Hamilton-Jacobi counterpart: more flexible to provide the corresponding model at a junction, with agents entering in incoming roads and exiting on outgoing roads.

Thanks for the attention

Microscopic Follow-the-Leader model

Let $N \in \mathbb{N}$, and consider a sample $0 \leq x^{N,0} \leq x^{N,1} \leq \ldots, x^{N,N} \leq L$ of $\bar{\rho}$:

$$\begin{cases} \bar{\rho}^N(x) = \frac{1}{N}\sum_{i=0}^{N-1} \frac{1}{x^{N,i+1}-x^{N,i}} 1_{[x^{N,i},x^{N,i+1})}(x) \rightharpoonup^* \bar{\rho}(x) \text{ in } L^\infty \\ x^{N,i+1} - x^{N,i} \geq \frac{1}{MN} \end{cases}$$

Let $t \mapsto X^{N,i}(t)$ for $1 = 0\ldots, N$ to be a solution to the Follow-the-Leader system:

$$(FtL) \begin{cases} \dot{X}^{N,N}(s) = v_{\max} & s \in (0,T) \\ \dot{X}^{N,i}(s) = v\left(\frac{1}{N(X^{N,i+1}(s)-X^{N,i}(s))}\right), & s \in (0,T), \quad i = 0,\ldots,N-1, \\ X^{N,i}(0) = x^{N,i} & i = 0,\ldots,N. \end{cases}$$

We associate to the trajectories of the FtL system the Eulerian density and empirical measure

$$\rho^N(t,x) = \frac{1}{N}\sum_{i=0}^{N-1} \frac{1_{[X^{N,i}(t),X^{N,i+1}(t))}(x)}{X^{N,i+1}(t)-X^{N,i}(t)} \qquad \hat{\rho}^N(t,dx) = \frac{1}{N}\sum_{i=0}^{N-1} \delta_{X^{N,i}(t)}.$$

It is by now classical the microscopic derivation of the LWR model through the discrete FtL model (ref: **Di Francesco, Rosini (ARMA 2015), Ancona, Bentaibi, Rossi (DCDS 2025)**): as $N \to +\infty$,

$$\rho^N(t,x) \to \rho(t,x) \qquad \text{in } L^1(0,T)\times(0,L),$$

$$\hat{\rho}^N(t,dx) \rightharpoonup^* \rho(t,x)\mathcal{L}^1 \text{ and in the topology } L^1_{loc}((0,T);W^1)$$

where ρ is the unique entropy weak solution of the (CL).