Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

The Turnpike phenomenon for Optimal Control Problems under Uncertainty

Michael Schuster, Noboru Sakamoto

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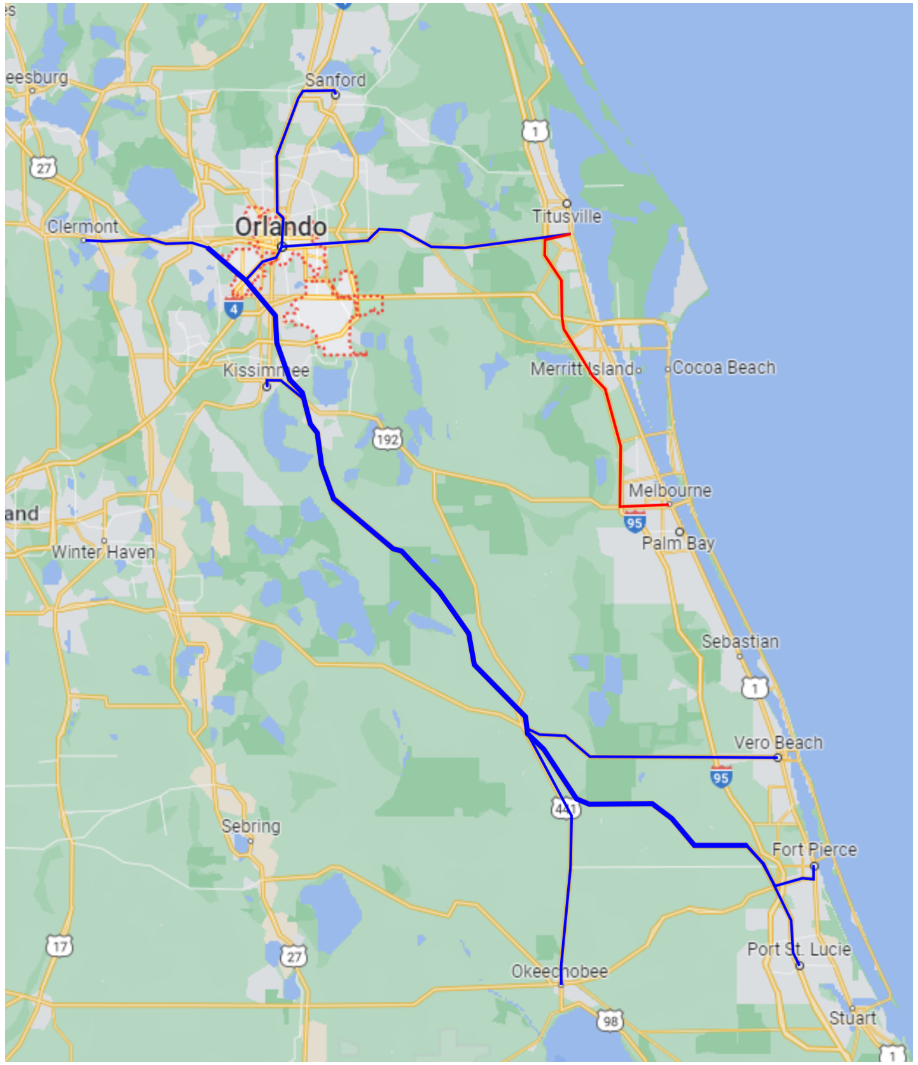
Friedrich-Alexander Universität Erlangen-Nürnberg (FAU),
Nanzan University Nagoya

Motivation

The Turnpike Phenomenon



- There is a fastest route between any two points and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
- But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

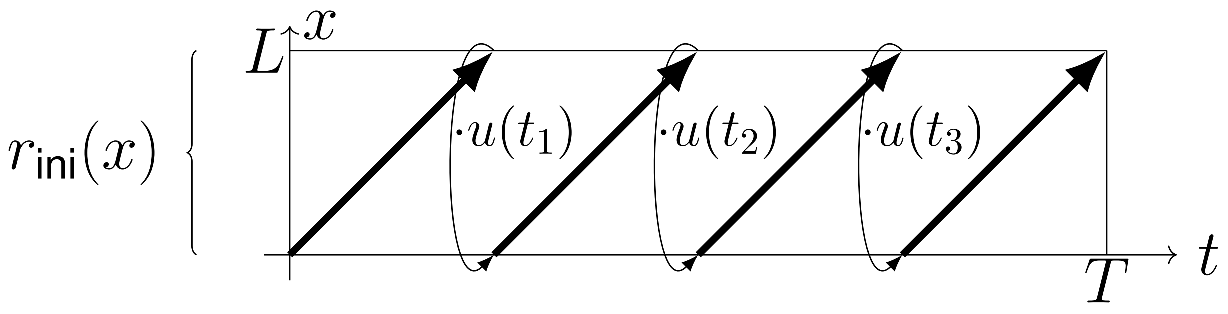
[Dorfman, Samuelson, Solow, 1958]: Linear Programming and Economic Analysis. *New York: McGraw-Hill*

Motivation

The Turnpike Phenomenon

Consider the optimal control problem governed by the transport equation with feedback control

$$
\begin{aligned}
\min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t), r(t,L)\big)\, dt \\
\text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\
& r(0,x) = r_{\text{ini}}(x), \\
& r(t,0) = g(u(t), r(t,L)),
\end{aligned}
$$



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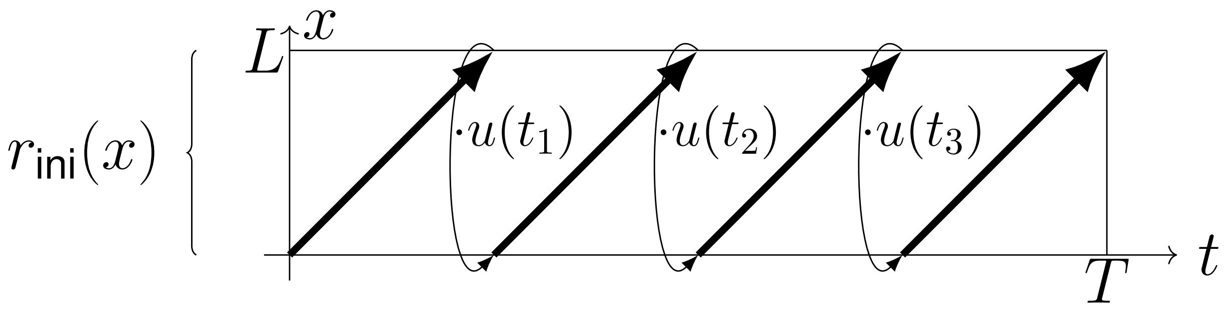
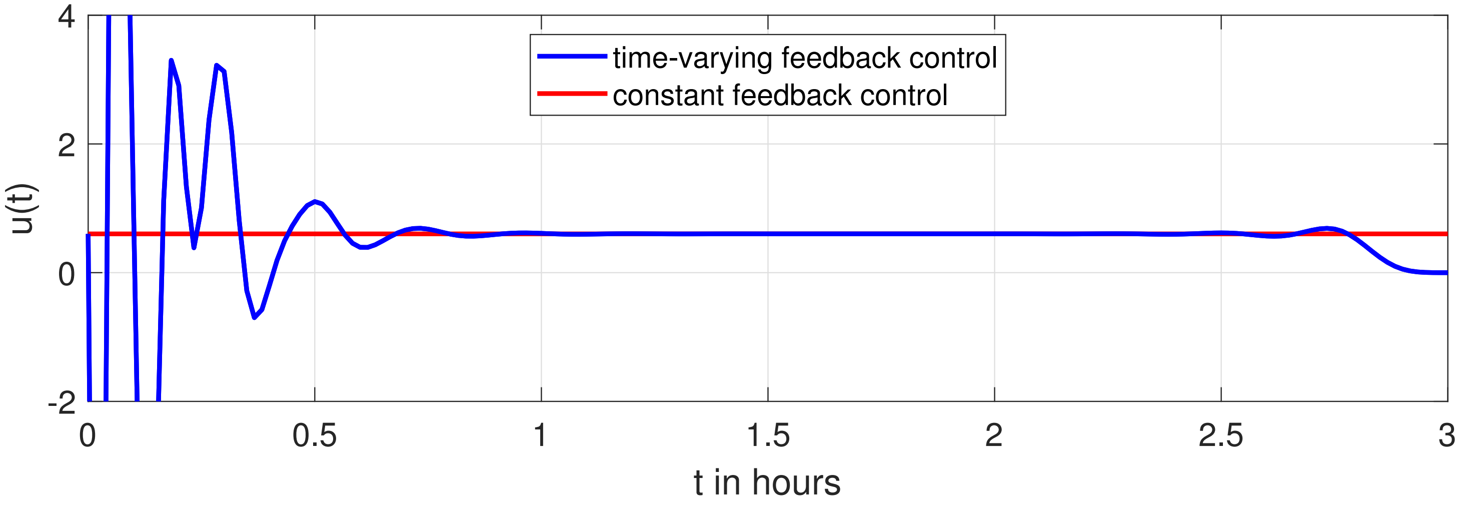




Motivation

The Turnpike Phenomenon

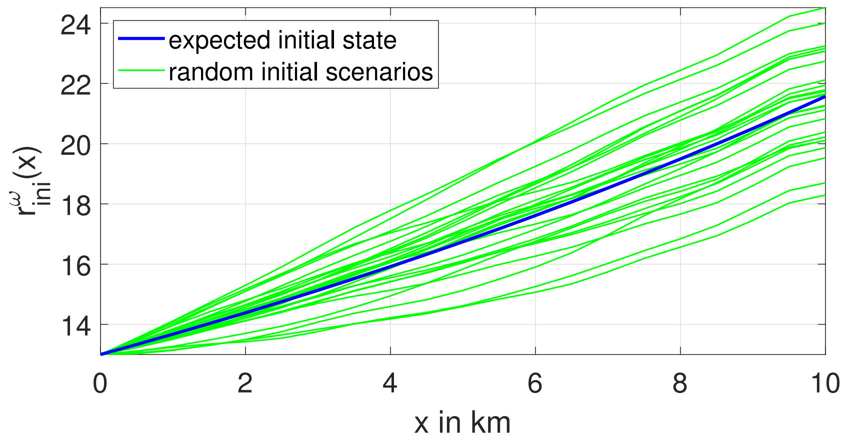
Consider the optimal control problem governed by the transport equation with random initial data, random source term and with feedback control

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\Big(u(t), \mathbb{E}\big[r(t,L)\big]\Big)\, dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m^\omega\, r(t,x),$$

$$r(0,x) = r^\omega_{\text{ini}}(x),$$

$$r(t,0) = g\Big(u(t), \mathbb{E}\big[r(t,L)\big]\Big),$$



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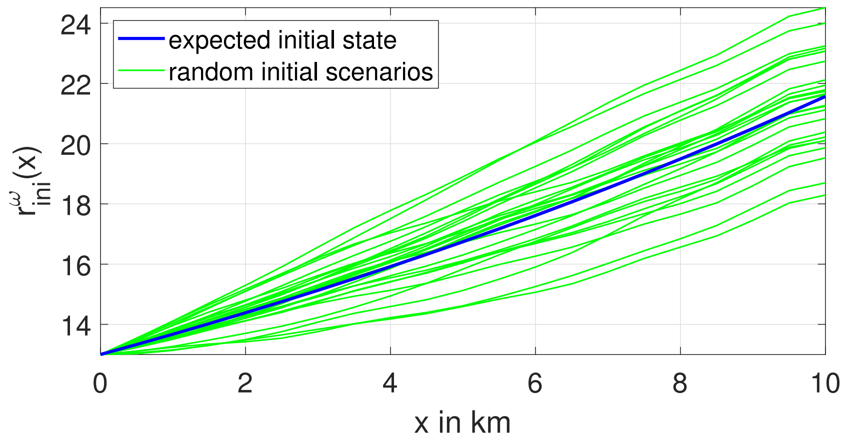

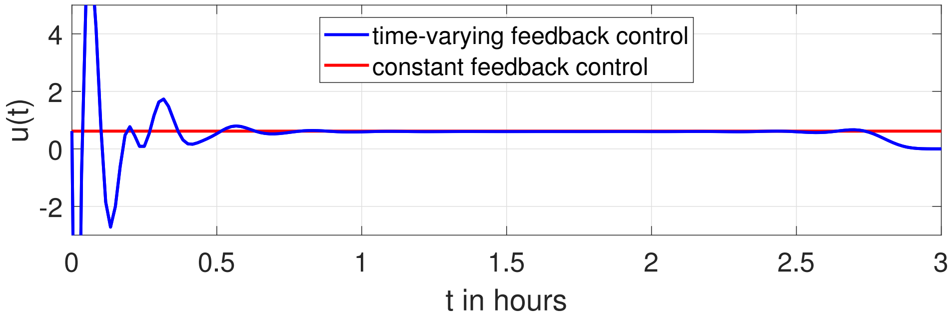

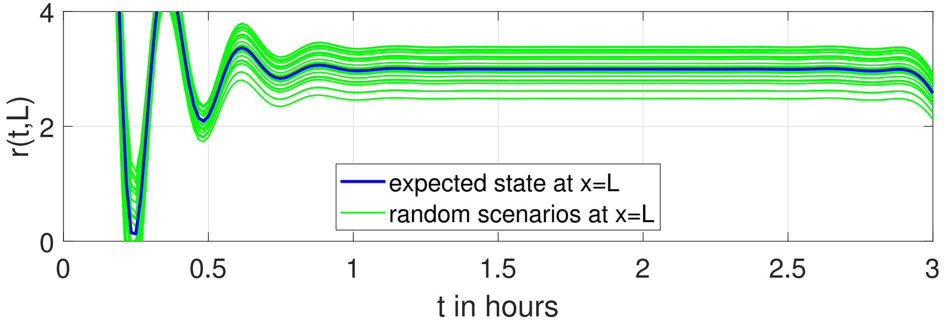

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

For $c > 0$ consider the transport equation in one dimension

$$r_t(t,x) + c\, r_x(t,x) = m\, r(t,x),$$

with initial condition and boundary control

$$r(0,x) = r_{\text{ini}}(x) \qquad \text{and} \qquad r(t,0) = u(t).$$

For convex functions f and g consider the optimal control problems

Dynamic Optimal Control Problem

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{aligned}$$

Static Optimal Control Problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m\, r(x), \\ & r(0) = u. \end{aligned}$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

(A1) For $\varepsilon > 0$ let functions f and g satisfy

$$\left(f'(x_1) - f'(x_2)\right)\left(x_1 - x_2\right) + \left(g'(y_1) - g'(y_2)\right)\left(y_1 - y_2\right) \;\geq\; \varepsilon \, \|x_1 - x_2\|_2^2.$$

(A2) Let the derivative of g be Lipschitz continuous with Lipschitz constant L_k, i.e.,

$$\|g'(y_1) - g'(y_2)\|_2 \;\leq\; L_k \, \|y_1 - y_2\|_2.$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

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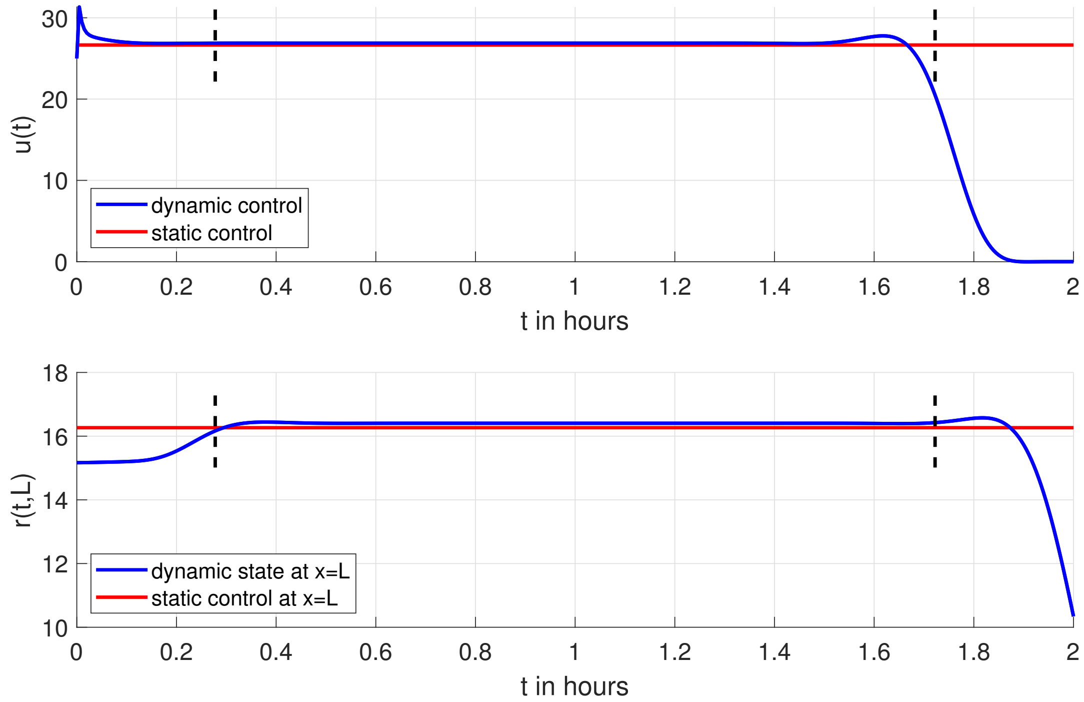
Theorem: Deterministic Turnpike

The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt$$
$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\, r(t,x),$$
$$r(0,x) = r_{\text{ini}}(x), \; r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \;\leq\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

Sketch of the proof.

Part I: The solution of the transport equation is given by

$$r(t,x) = \begin{cases} \exp(\ mt\ )\ r_{\text{ini}}(\ x - ct\ ) & x > ct \\ \exp\left(\ m\frac{x}{c}\ \right)\ u\left(\ t - \frac{x}{c}\ \right) & x \leq ct \end{cases}$$

Part II: The derivative of the objective function is given by

$$J_T'(u) = f'\big(u(t)\big) + k\ g'\big(k\ u(t)\big)\ \psi(t),$$

with $k = \exp\left(m\frac{L}{c}\right)$ and $\psi(t) = \begin{cases} 1 & 0 < t < T - \frac{L}{c} \\ 0 & \text{else} \end{cases}$.

Part III: Let $u^\delta(t)$ and u^σ be the optimal dynamic and static solution. Then we have necessary optimality condition

$$f'\big(u^\delta(t)\big) - f'\big(u^\sigma\big) = k\ g'\big(k\ u^\sigma\big) - k\ g'\big(k\ u^\delta(t)\big)\ \psi(t).$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

Sketch of the proof.

Part IV: Starting from assumption (A1) we have

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \;\leq\; \int_0^T \Big(f'(u^\delta(t)) - f'(u^\sigma)\Big)\Big(u^\delta(t) - u^\sigma\Big) + \Big(g'(r^\delta(t,L)) - g'(r^\sigma(L))\Big)\Big(r^\delta(t,L) - r^\sigma(L)\Big) \, dt$$

$\vdots$

$\vdots$ (apply necessary optimality conditions, use integration by substitution)

$\vdots$

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \;\leq\; \int_{T-\frac{L}{c}}^T k\, g'(k\, u^\sigma)\Big(u^\delta(t) - u^\sigma\Big) \, dt + \int_0^{\frac{L}{c}} \Big(g'\big(\exp(mt) r_{\text{ini}}(L - ct)\big) - g'(ku^\sigma)\Big)\Big(\exp(mt) r_{\text{ini}}(L - ct) - ku^\sigma\Big) \, dt$$

$\vdots$

$\vdots$ (apply Cauchy-Schwarz inequality and (A2))

$\vdots$

$$\varepsilon \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 \;\leq\; z_1 \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} + \frac{L_k}{c} z_2^2 \, \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3$$

□

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

Assume that the initial data is given by a randomized *Fourier series*, i.e., for a famiy of identically distributed random variables, we have

$$a_m(r_{\mathsf{ini}}) := \int_0^L r_{\mathsf{ini}}(x)\psi_m(x)\, dx, \qquad \psi_m(x) := \frac{\sqrt{2}}{\sqrt{L}}\left(\left(\frac{\pi}{2} + m\pi\right)\frac{x}{L}\right), \qquad r_{\mathsf{ini}}^{\omega}(x) = \sum_{m=0}^{\infty} \xi_m(\omega)\, a_m(r_{\mathsf{ini}})\, \psi_m(x).$$

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Theorem: TP with uncertain initial data

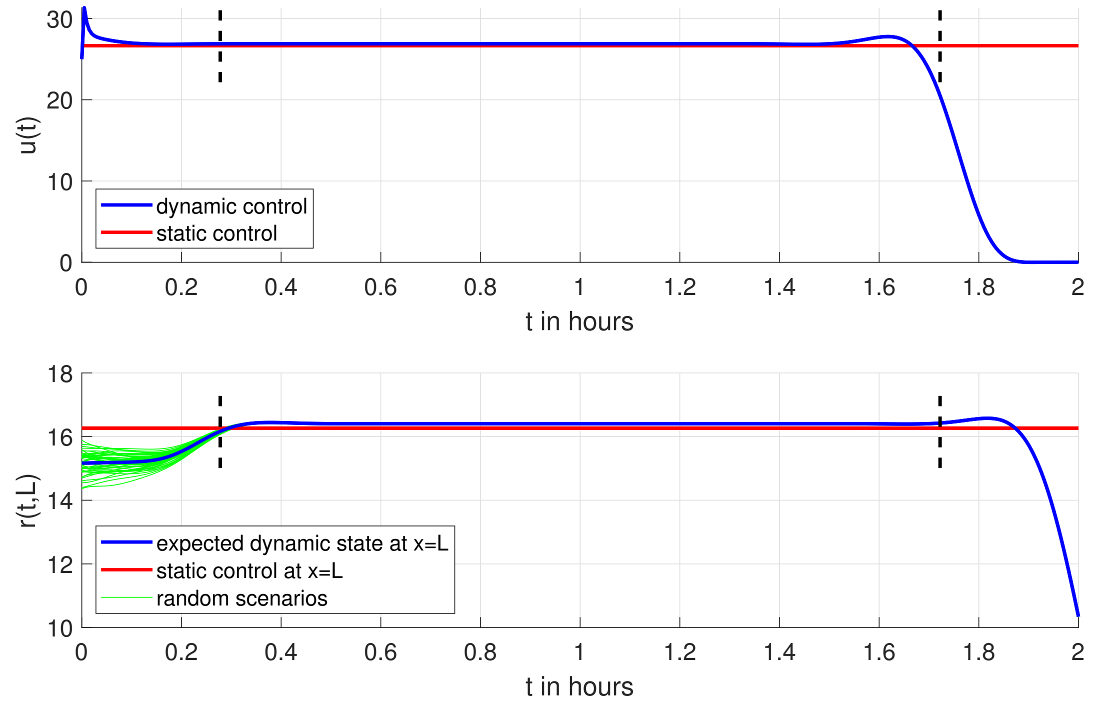
The optimal solution $u^{\delta}(t)$ of the dynamic optimal control problem

$$\min_{u\in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\,dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\, r(t,x), \quad r(0,x) = r^{\omega}_{\text{ini}}(x),\; r(t,0) = u(t).$$

and the optimal solution u^{σ} of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^{\delta}(t) - u^{\sigma}\|_2^2\,dt \;\le\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

We randomize the source term by a random variable ξ on an appropriate probability space: $m^\omega := \xi(\omega), \quad \omega \in \Omega$

(A3) Assume that $e_0(t)$ is uniformly bounded, where e_0 is defined as

$$e_0 : [0,T] \to \mathbb{R} \cup \{\pm\infty\} \quad t \mapsto \int_{-\infty}^{\infty} \exp(zt)\, \varrho_\xi(z)\, dz,$$

(A4) Assume that

$$\int_{-\infty}^{\infty} \exp\left(z\frac{L}{c}\right) \varrho_\xi(z)\, dz \quad < \infty$$

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Theorem: TP with randomized source term

The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

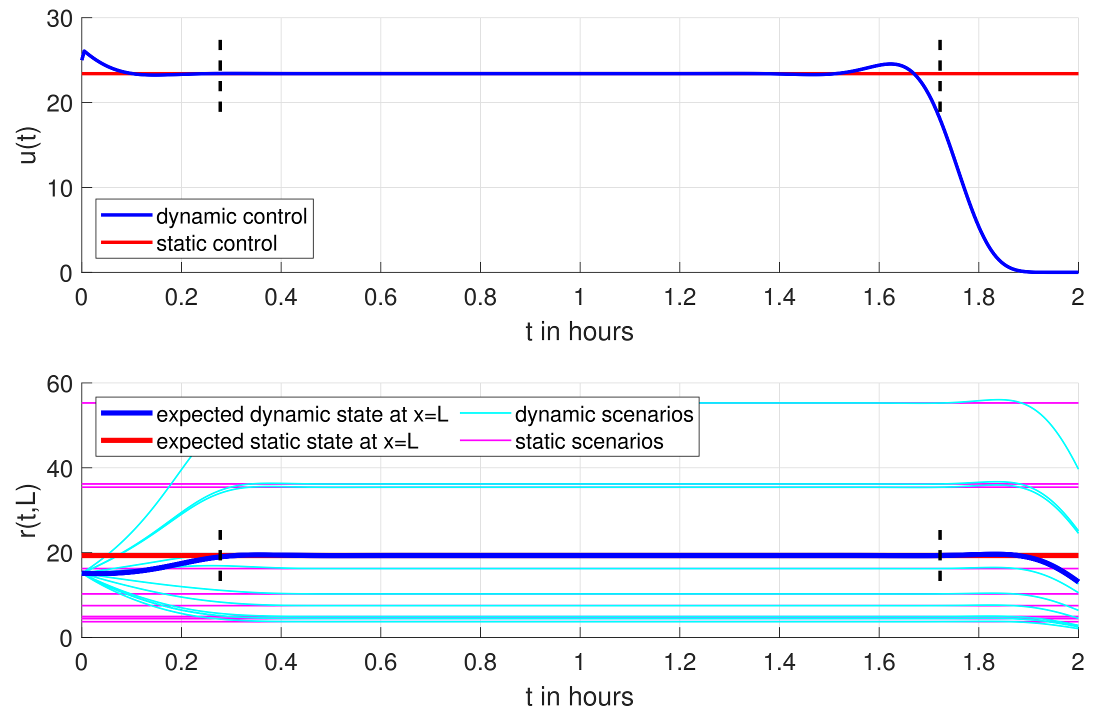
$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\, dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m^\omega\, r(t,x),$$

$$r(0,x) = r_{\text{ini}}(x), \; r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\, dt \;\leq\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

Sketch of the proof.

Part I: Compute the solution $r^\omega(t,x)$ of the transport equation

Part II: Compute the expected value in the objective function:
For a random variable X and a function h we have $\mathbb{E}\big(h(X)\big) = \int_{-\infty}^{\infty} h(z)\,\varrho_X(z)\,dz$. Thus we have

$$\begin{aligned} J_T(u) &= \int_0^T f(u(t))\,dt + \int_0^T g\Big(\mathbb{E}\big(r^\omega(t,x)\big)\Big)\,dt \\ &= \int_0^T f(u(t))\,dt + \int_0^{\frac{L}{c}} g\bigg(r_{\text{ini}}(L-ct)\int_{-\infty}^{\infty}\exp(zt)\,\varrho_\xi(z)\,dz\bigg)\,dt + \int_{\frac{L}{c}}^T g\bigg(u\Big(t-\frac{L}{c}\Big)\int_{-\infty}^{\infty}\exp\Big(z\frac{L}{c}\Big)\,\varrho_\xi(z)\,dz\bigg)\,dt \\ &= \int_0^T f(u(t))\,dt + \int_0^{\frac{L}{c}} g\big(e_0(t)\,r_{\text{ini}}(L-ct)\big)\,dt + \int_{\frac{L}{c}}^T g\bigg(e_1\,u\Big(t-\frac{L}{c}\Big)\bigg)\,dt \end{aligned}$$

Part III: Compute the derivative of the objective function

Part IV: State the necessary optimality conditions

Part V: Get the Turnpike estimate by applying (A1) and (A2) □

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

A turnpike inequality also holds for the expected values, i.e., under the assumptions of the last theorem we have:

$$\int_0^T \left\| \mathbb{E}\left[r^\delta(t,x)\right] - \mathbb{E}\left[r^\sigma(x)\right] \right\|_2^2 \, dt \leq \tilde{C} \qquad \forall x \in [0,L].$$

Since the variances depend on the squared expected states, a stronger assumption is necessary to guarantee the convergence of the squared controls:

(A5) For $\varepsilon > 0$ let functions f and g satisfy

$$\left(f'(x_1) - f'(x_2)\right)\left(x_1 - x_2\right) + \left(g'(y_1) - g'(y_2)\right)\left(y_1 - y_2\right) \;\geq\; \varepsilon \, \|x_1^2 - x_2^2\|_2^2.$$

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

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Theorem: TP for the variances

Under the assumptions (A2) - (A5), the optimal controls $u^\delta(t)$ and u^σ satisfy the turnpike inequality

$$\int_0^T \left\|(u^\delta(t))^2 - (u^\sigma)^2\right\|_2^2 dt \leq \mathcal{C},$$

and for the variances of the corresponding optimal states we have

$$\int_0^T \left\|\mathrm{Var}\left[r^\delta(t,x)\right] - \mathrm{Var}\left[r^\sigma(x)\right]\right\|_2^2 dt \leq \tilde{\mathcal{C}}.$$

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

We have the convergence results

$$\lim_{T\to\infty}\int_0^T r^\delta(t,x,\omega)\,dt = r^\sigma(x,\omega), \qquad \lim_{T\to\infty}\int_0^T \mathbb{E}\big[r^\delta(t,x)\big] = \mathbb{E}\big[r^\sigma(x)\big], \qquad \lim_{T\to\infty}\int_0^T \mathrm{Var}\big[r^\delta(t,x)\big] = \mathrm{Var}\big[r^\sigma(x)\big].$$

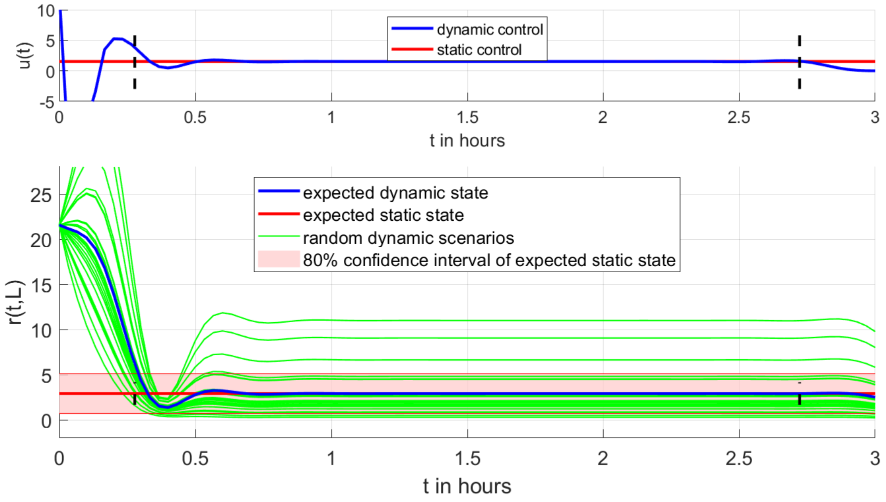
A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

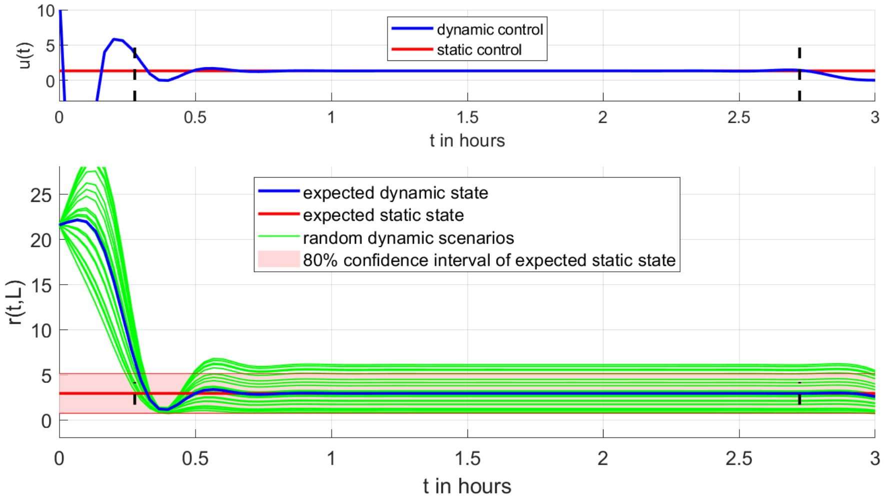
We have the convergence results

$$\lim_{T\to\infty}\int_0^T r^\delta(t,x,\omega)\,dt = r^\sigma(x,\omega), \qquad \lim_{T\to\infty}\int_0^T \mathbb{E}\big[r^\delta(t,x)\big] = \mathbb{E}\big[r^\sigma(x)\big], \qquad \lim_{T\to\infty}\int_0^T \mathrm{Var}\big[r^\delta(t,x)\big] = \mathrm{Var}\big[r^\sigma(x)\big].$$

$\Rightarrow$ The 80% confidence interval around the expected steady state $\mathbb{E}\big[r^\sigma(x)\big]$ also contains 80% of the dynamic random scenarios.



(a) Gaussian distributed random variable



(b) Uniformly distributed random variable

A Turnpike Result for the Transport Eq.

Non-Autonomous and Space-Variant Tp Eq

Consider the optimal control problem governed by an non-autonomous and space-variant transport equation

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m(x)\, r(t,x) + b(x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t), \end{aligned}$$

and its corresponding static problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m(x)\, r(x) + b(x), \\ & r(0) = u. \end{aligned}$$

$\Rightarrow$ We can get an integral Turnpike result following the proof of the previous results.

A Turnpike Result for the Transport Eq.

Space- and Time-Variant Transport Equation

Consider the optimal control problem governed by an non-autonomous, space- and time-variant transport equation

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m(t,x)\, r(t,x) + b(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t), \end{aligned}$$

$\Rightarrow$ Numerical simulations also provide a turnpike result, but the Turnpike is not given by a steady state problem but by the corresponding optimal control problem with constant control.

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m(t,x)\, r(t,x) + b(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u, \end{aligned}$$

A Turnpike Result for the Transport Eq.

Optimal Control with Feedback Control

Consider the optimal control problem with feedback control from the beginning:

$$
\begin{aligned}
\min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t), r(t,L)\big)\, dt \\
\text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\
& r(0,x) = r_{\text{ini}}(x), \\
& r(t,0) = u(t) \cdot r(t,L),
\end{aligned}
$$

A solution of the transport system is given by

$$
r_k(t,x) = \begin{cases} r_{\text{ini}}(kL + x - ct)\ \exp(mt) \displaystyle\prod_{i=0}^{k-1} u\Big(t - \tfrac{iL+x}{c}\Big) & kL \leq ct < kL + x \\ r_{\text{ini}}\big((k+1)L + x - ct\big)\ \exp(mt) \displaystyle\prod_{i=0}^{k} u\Big(t - \tfrac{iL+x}{c}\Big) & kL + x \leq ct < (k+1)L \end{cases}
$$

⇒ The turnpike is given by the corresponding optimal control problem with constant feedback control, but the proof is not clear.

A Turnpike Result for the Transport Eq.

A Semi-Linear Transport Equation

Consider an optimal boundary control problem with the transport equation nonlinear source term:

$$\begin{aligned}
\min_{u\in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\
\text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = h\big(r(t,x)\big), \\
& r(0,x) = r_{\text{ini}}(x), \\
& r(t,0) = u(t).
\end{aligned}$$

A solution of the transport equation with nonlinear source term is given by

$$r(t,x) = \begin{cases} G^{-1}\Big(t + G\big(r_{\text{ini}}(x-ct) \big)\Big) & x > c\,t, \\ G^{-1}\Big(\frac{x}{c} + G\big(u(t - \frac{x}{c}) \big) \Big) & x \leq c\,t, \end{cases}$$

where G is an anti-derivative of $1/h$.

$\Rightarrow$ Numerical simulations also provide a turnpike result, but due to the nonlinear dependence of the control we cannot directly get the derivative (**work in progress**).

Optimal Boundary Control for the Transport Equation under Uncertainty

[Sakamoto and Schuster, 2025]: *A Turnpike Result for Boundary Control Problems governed by the Transport Equation under Uncertainty.* Under revision, Preprint available
https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/509